ANALYSIS OF WIDEBAND PHASE-SHIFT NETWORKS

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LIST OF SYMBOLS

Symbol	Definition
j	$\sqrt{-1}$
f	Frequency in cycles per second
ω	$2\pi f$
p	$j\omega$
F	$\mathrm{Log}_{10} f$
θ	Phase angle
s	Parameter of phase angle equation
V_{34}	Phase-shift section output voltage
V_{13}, V_{23}	Phase-shift section driving voltages
A_o	Network reduction constant
$\cong$	Approximately equal to

Subscripts	
M	Center frequency
A	Channel A
B	Channel B
D	Dome's method
S	Slope method

LIST OF FIGURES

ANALYSIS OF WIDEBAND PHASE-SHIFT NETWORKS

INTRODUCTION

Since the time Volta discovered that electric current would flow in a closed loop of wire when connected to an elementary cell, man has utilized electric circuits to perform specified tasks. Exclusive of any transient effects, circuits energized with direct current cause only diminution of strength, or attenuation, of a transmitted voltage. Circuits energized with alternating current, on the other hand, not only are subject to attenuation, but, in addition, may cause the phase relationship between current and voltage at the source to be different from that at the load. When the frequency of the applied alternating current is constant, the phase relationships remain constant, because the circuit offers unvarying impedance to the flow of current. However, the transmission of speech or intelligence involves the use of many frequencies. The reactive component of impedance varies with frequency and thus will cause unlike phase shifts at different frequencies.

In the more general types of four-terminal networks, attenuation and phase shift become functions of frequency. Changing attenuation usually occurs when the phase angle is substantially constant; a low, changing value of attenuation occurs when the phase angle is varying. Under this condition, the attenuation is zero in the case of purely reactive networks. The phase shift and attenuation are inherently linked together in this type of network. In certain applications, it is desired to control the attenuation and phase shift angle independently of each other. It is readily seen that special networks must be devised to achieve this.

Two, equal output voltages differing in phase by 90° over a wide frequency range are required in some single-sideband transmission systems and in the production of circular cathode ray oscilloscope traces over wide frequency ranges. Each of the output voltages must then satisfy the requirements of constancy in magnitude over the frequency range and yet retain a constant phase difference of 90°, in absolute value, with respect to the other.

In particular, a single-sideband transmitter described in a recent publication employs networks in which the calculated difference in phase is held constant at 90±.2° and the calculated output held constant to a relative value of 1±.285 over that portion of the audible frequency range from 300 to 3000 cycles per second.[1] The variation from an exact 90° phase shift and constant output is compensated for in other portions of the single-sideband generating system. Lenehan[2] has described a somewhat similar system of producing the desired phase difference and output over approximately the same frequency range. Three variations of still another method of obtaining the desired phase and amplitude characteristics are described by Dome[3].

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[1] Honnell, M. A., "Single-Sideband Generator", Electronics, November, 1945, pp. 166-168.

[2] Lenehan, B. E., "A New Single-Sideband Carrier System", Electrical Engineering, June, 1947, pp. 549-552.

[3] Dome, R. B., "Wideband Phase-Shift Networks", Electronics, December, 1946, pp. 112-115.

With the latter method, a $90 \pm 4^\circ$ phase difference is obtained over the frequency band from 130 to 3000 cycles per second. While the phase characteristic is not quite as desirable as that obtained in some of the other systems, Dome points out that a more critical choice of circuit parameters will reduce the variation from a true 90° phase shift. Moreover, this latter method has one distinct advantage, the output remains constant over the desired frequency range.

It is the purpose of this thesis to analyze the three types of networks described by Dome, to extend the range of two and to construct an experimental model based on the results of the extension.

GENERAL DISCUSSION OF NETWORKS AND PHASE SHIFT SYSTEMS

A general form of the networks chosen by Dome is shown in Figure 1. Since it possesses the structure of a lattice network (see Figure 2), but has only half the number of impedance arms, it is designated a "semi-lattice" network. Zero-impedance voltage sources, equal in magnitude to each other, but of opposite polarity, make up the remaining arms of the lattice. The lattice type has phase and amplitude characteristics, such that one may be specified independently of the other, which fact makes the lattice ideally suited for the purpose.[4]

In order to utilize a full lattice, it would be necessary to employ an ideal transformer as an impedance matching device between the generator and load. Such an arrangement is shown in Figure 3. The half-lattice structure avoids the use of transformers and yet retains the desired phase and amplitude characteristics. A general solution for the output voltage of the half-lattice structure, in terms of the circuit impedances and the input voltages, is contained in Appendix I.

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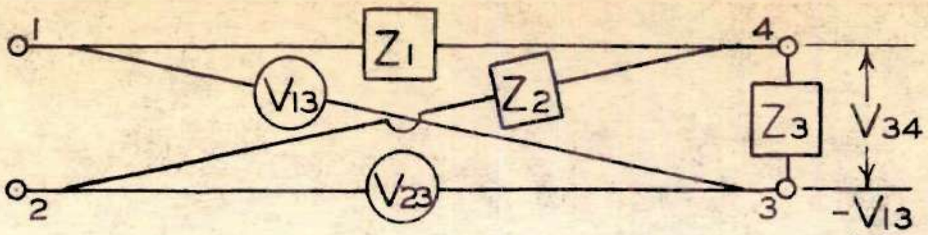
[4]Terman, F. E., Radio Engineers' Handbook (New York, McGraw-Hill Book Co., Inc., 1943) p. 238.

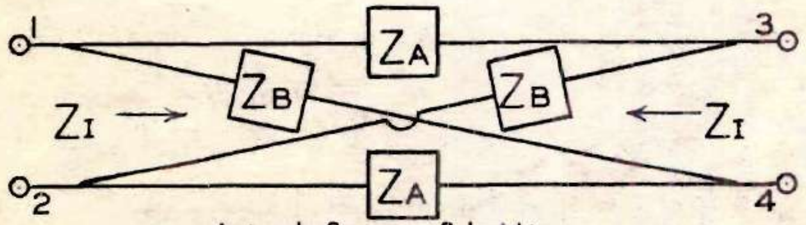


$$-V_{13} = V_{23}$$

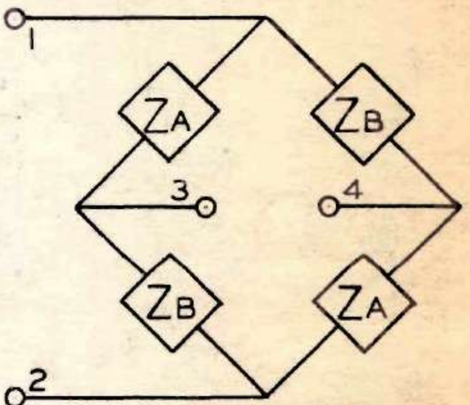
$$V_{34} = V_{13}\frac{(Z_1 - Z_2)}{Z_1 Z_2/Z_3 + (Z_1 + Z_2)}$$

SEMI-LATTICE STRUCTURE

FIG. 1



Mesh form of lattice



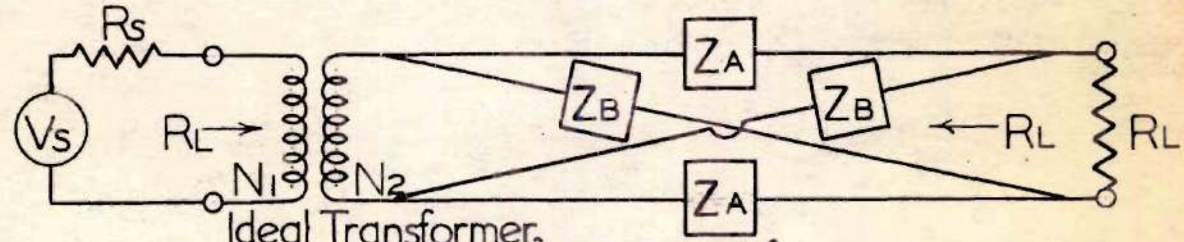
Bridge form of lattice

$Z_I = \sqrt{Z_A Z_B}$, Z_I is Image Impedance.

$\Theta = 2\tan^{-1}\sqrt{Z_A/Z_B}$, Θ is Image Transfer Constant.

LATTICE STRUCTURE

FIG. 2



Ideal Transformer,
$R_S/R_L = (N_1/N_2)^2 = (\text{Turns ratio})^2$

$Z_A Z_B = R_L^2$. Z_A and Z_B are inverse complex impedances such their product is equal to a real quantity, R_L^2, at all frequencies.

IMAGE MATCH BETWEEN SOURCE AND LOAD BY MEANS OF IDEAL TRANSFORMER

FIG. 3

PART I

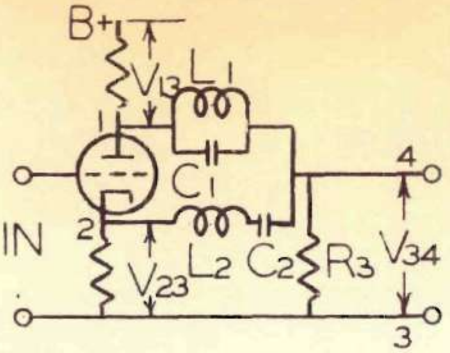
ANALYSIS OF NETWORKS

ANALYSIS OF ONE POLE L-C NETWORK

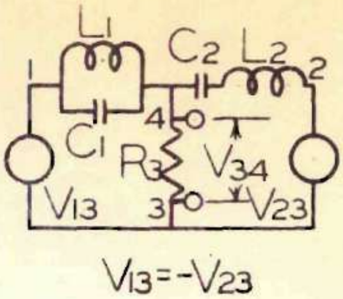
The first network to be considered contains purely reactive elements in combination with a single resistance as the output load. A schematic diagram of this network is shown in Figure 4. The actual use of such a network is limited to the use of high-Q reactive elements. However, the networks of pure inductances and capacitances serve one extremely useful purpose. They may be used as guides to a more practical solution. It is chiefly with this end in mind that the network of Figure 4 is analyzed.

Two networks, each identical to that shown in Figure 4, except for the values of the circuit elements in the phase-shift section, comprise two channels, A and B. References to similar quantities in each network are identified by the subscript A or B following the quantity in question. When energized with a common voltage source of variable frequency, channels A and B each cause a different phase shift between the common input voltage and the voltages V_{34A} and V_{34B}. By so adjusting the constants of the phase-shift sections, V_{34A} and V_{34B} retain a phase difference of approximately 90° over a wide frequency range and remain equal in absolute value.

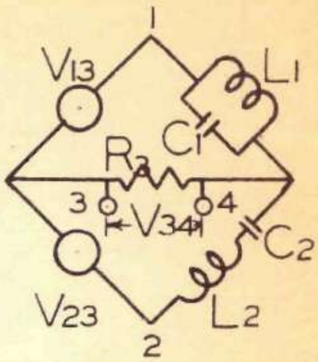
In discussing one channel, it may be seen that with equal plate and cathode resistors, V_{13} is equal and opposite in polarity to V_{23}, since a common current flows through the resistors. For this analysis,



Schematic arrangement of single network.



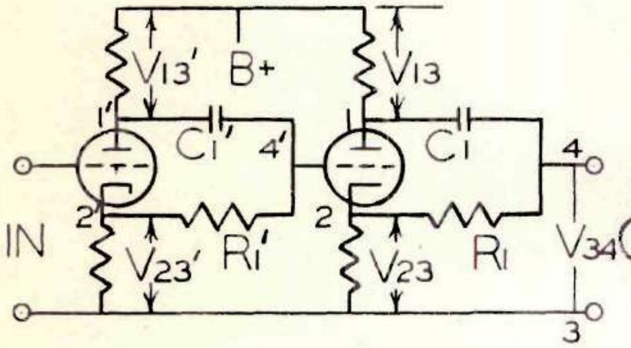
Mesh form of single network.



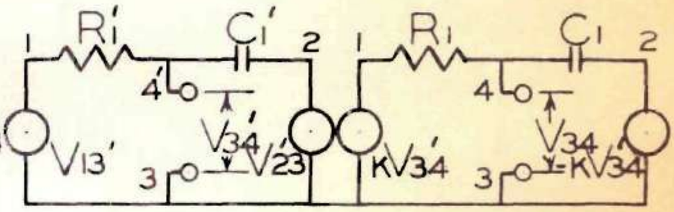
Bridge form of single network.

ONE POLE L-C NETWORK

FIG. 4



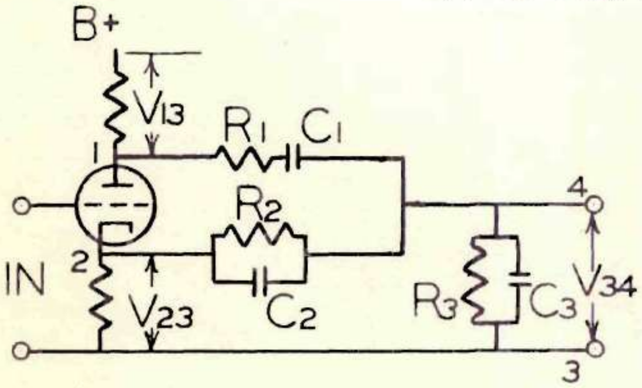
Schematic arrangement of single network.



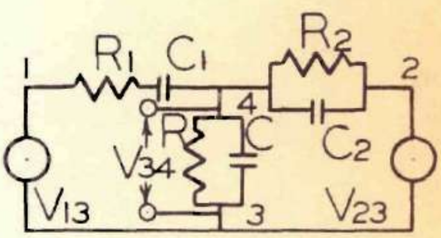
Mesh form of single network.

R-C NETWORK

FIG. 5



Schematic arrangement of single network.



Mesh form of single network.

R-C NETWORK

FIG. 6

it is assumed that the impedance of the power supply is practically zero for alternating currents and that the impedance of the cathode and plate resistors is negligible compared to R_3.

It will be demonstrated that over a predetermined range of frequencies, the phase angle of V_{34} varies almost directly with the common logarithm of frequency, $\log_{10}f$. In all following developments, $\log_{10}f$ may be designated by F, and f by 10^F. The relationships indicated may be used interchangeably. The equation which governs the relationship between F and the phase angle, θ, of V_{34} with respect to V_{13} is the equation of the tangent of θ versus F. Over a central range of frequencies, the following equations determine, approximately, the phase angles in terms of $\log_{10}f$ or F:

$$-\theta_A \cong m \log_{10} f + K_A \qquad (1a)$$
$$-\theta_B \cong m \log_{10} f + K_B \qquad (1b)$$

where m is a slope in degrees per unit of F and K_A, K_B are intercepts on the $F = 0$ axis. The minus sign associated with the phase angles indicates that the angles increase negatively as the frequency is increased. It should be pointed out that while certain curves of $\log_{10}f$ versus θ are plotted with θ increasing in an upward direction, the increase in value of θ is in reality a negative increase in the value of θ. Unless otherwise specified, θ is henceforth to be considered a negative quantity.

An entire curve of $\log_{10}f$ versus θ is symmetrical about some value $F = F_n = \log_{10}f_n$. This fact will be demonstrated later, but for the present, the general equation

$$\theta = G(f) = H(F) \tag{2}$$

where G and H represent functional relationships will suffice. The region of a curve where

$$\theta = mF + K \tag{3}$$

is also symmetrical about F_n. One of the desired results, a phase difference of approximately 90° between the two output voltages, is obtained if F_{nA} and F_{nB} are chosen such that any F in the common range of the "linear" portions of the phase curves for networks A and B determines

$$\theta_A - \theta_B \cong 90^{\circ} \tag{4}$$

when F is substituted in equations

$$\theta_A \cong mF + K_A \tag{5a}$$
$$\theta_B \cong mF + K_B . \tag{5b}$$

Appendix I illustrates that

$$V_{34} = V_{13} \frac{(Z_1 - Z_2)}{\frac{Z_1 Z_2}{Z_3} + (Z_1 + Z_2)} . \tag{6}$$

In the network of Figure 4,

$$Z_1 = \frac{pL_1}{(1 + p^2L_1C_1)} \tag{7}$$

$$Z_2 = \frac{(1 + p^2L_2C_2)}{pC_2} \tag{8}$$

$$Z_3 = R_3 \tag{9}$$

where $p = j\omega = j2\pi f$. For further simplicity, if $L_1C_1 = a$, $L_2C_2 = b$ and $L_1C_2 = d$, equation (6) becomes

$$V_{34} = V_{13}\,\frac{\left[-p^4ab + p^2(d-b-a) - 1\right]}{\left[p^4ab + p^3bL_1/R_3 + p^2(d+b+a) + pL_1/R_3 + 1\right]} \tag{10}$$

which then reduces to

$$V_{34} = V_{13}\,\frac{\left[-\omega^4ab - \omega^2(d-b-a) - 1\right]}{\left[\omega^4ab - \omega^2(d+b+a) + 1\right] - j\left[\omega^3bL_1/R_3 - \omega L_1/R_3\right]}. \tag{11}$$

The phase angle associated with V_{34} is the phase angle of the coefficient of V_{13} in equation (11).

Since V_{13} is the reference voltage, it contributes nothing to the total phase angle. Further, with suitable restrictions, the magnitude of the above expression is equal to unity. The modulus or magnitude of a complex number is equal to the square root of the sum of the squares of the real and imaginary parts.[5] Hence, the re-

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[5] Sokolnikoff, I. S. and E. S. Sokolnikoff, Higher Mathematics for Engineers and Physicists (New York, McGraw-Hill Book Co., Inc., 1941) p. 442.

strictions which must be imposed are

$$1 = \frac{[-\omega^4 ab - \omega^2(d-b-a) - 1]}{\sqrt{[\omega^4 ab - \omega^2(d+b+a)+1]^2 + [\omega^3 bL_1/R_3 - \omega L_1/R_3]^2}} \quad (12)$$

or, after squaring each side, transposing and arranging terms in descending powers of ω,

$$\omega^8(a^2b^2 - a^2b^2) + \omega^6(-4ab^2 - 4a^2b + b^2L_1^2/R_3^2) + \omega^4(4ab + 4bd + 4ad - 2bL_1^2/R_3^2) + \omega^2(-4d + L_1^2/R_3^2) = 0. \quad (13)$$

In order for the magnitude to be independent of frequency, each coefficient of the powers of ω must be equal to zero. Consideration of the coefficient of ω^2 yields

$$4d = L_1^2/R_3^2 \; ; \quad L_1^2/d = 4R_3^2. \quad (14)$$

By substituting for d its value, L_1C_2, equation (14) becomes

$$L_1/C_2 = 4R_3^2. \quad (15)$$

Evaluation of the coefficient of the ω^4 term gives

$$4ab + 4bd + 4ad = 2bL_1^2/R_3^2 \quad (16)$$

and when values for a, b and d are substituted,

$$\frac{2L_1L_2}{C_1L_2 + C_2L_2 + C_1L_1} = 4R_3^2. \quad (17)$$

The coefficient of the ω^6 term when equated to zero is

$$4ab^2 + 4a^2b = b^2L_1^2/R_3^2 \qquad (18)$$

or

$$\frac{L_1L_2C_2}{C_1(L_2C_2 + L_1C_1)} = 4R_3^2. \qquad (19)$$

By combining equations (15) and (17)

$$\frac{L_1}{C_2} = \frac{2L_1L_2}{C_1L_2 + C_2L_2 + C_1L_1} \qquad (20)$$

or

$$C_1L_2 + C_2L_2 + C_1L_1 = 2L_2C_2 \qquad (21)$$

and

$$L_1C_1 = L_2C_2 - L_2C_1. \qquad (22)$$

After dividing each side by C_1C_2, equation (22) becomes

$$L_1/C_2 = L_2/C_1 - L_2/C_2 \qquad (23)$$

or

$$L_1/C_2 = L_2/C_1(1 - C_1/C_2). \qquad (24)$$

By choosing C_1 and C_2 such that C_1/C_2 is small,

$$L_1/C_2 \cong L_2/C_1 \; ; \; L_1C_1 \cong L_2C_2. \tag{25}$$

With further reference to equation (11), if θ is the angle associated with the coefficient of V_{13}, $-\theta$ will then be the angle of the reciprocal of the coefficient. It follows that

$$-\theta = \tan^{-1} \frac{[-\omega^3 bL_1/R_3 + \omega L_1/R_3]}{[\omega^4 ab - \omega^2(d+b+a) + 1]} \tag{26}$$

and

$$\theta = \tan^{-1} \frac{[\omega^3 b L_1/R_3 + \omega L_1/R_3]}{[\omega^4 ab - \omega^2(d+b+a) + 1]}. \tag{27}$$

At a value of $\omega = \omega_o$, the impedance of Z_1 becomes infinite. $\omega_o/2\pi$ is the anti-resonant frequency of the parallel L-C combination which constitutes Z_1. This anti-resonant frequency may be evaluated in terms of L_1C_1 as follows:

$$\omega_o/2\pi = f_o = 1/2\pi \sqrt{\frac{1}{L_1C_1}}. \tag{28}$$

Further, the resonant frequency of a series L-C combination is also equal to $1/2\pi$ divided by the square root of the product of the capacitance (in farads) and the inductance (in henries). At the resonant frequency, the impedance of the L-C combination is zero. The fact that $L_1C_1 \cong L_2C_2$ makes it possible to allow

$$f_o \cong 1/2\pi\sqrt{L_2C_2}\,. \qquad (29)$$

By letting $L_1/L_2 = s^2$, it is seen that for s sufficiently large, the difference between L_1C_1 and L_2C_2 is negligible. For an s of 5, the difference between L_1C_1 and L_2C_2 is four percent of L_1C_1; for an s of 10, the difference is one percent of L_1C_1.

The following evaluation of the constants a, b, d and s may now be made:

$$\begin{aligned} a &= L_1C_1 = 1/\omega_o^2 \\ b &= L_2C_2 \cong 1/\omega_o^2 \\ d &= L_1C_2 = C_2/C_1 \cdot L_1C_1 \cong s^2/\omega_o^2 \\ s^2 &\cong C_2/C_1 \cong L_1/L_2\,. \end{aligned} \qquad (30)$$

From equation (15),

$$L_1/C_2 = 4R_3^2\,. \qquad (15)$$

Then

$$\theta = \tan^{-1}\frac{L_1/R_3(\omega^3/\omega_o^2 - \omega)}{(\omega^4/\omega_o^4 - 2\omega^2/\omega_o^2 + 1) - s^2\omega^2/\omega_o^2}\,. \qquad (31)$$

After substituting the value for R_3 and clearing the ω_o terms from the denominator of the fraction

$$\theta = \tan^{-1} \frac{2\sqrt{L_1C_2}\,(\omega^3\omega_o^2 - \omega_o^4\omega)}{(\omega^4 - 2\omega^2\omega_o^2 + \omega_o^4) - S^2\omega^2\omega_o^2} \tag{32}$$

or

$$\theta = \tan^{-1} \frac{2s\,(\omega^3\omega_o - \omega\omega_o^3)}{(\omega^4 - 2\omega^2\omega_o^2 + \omega_o^4) - S^2\omega^2\omega_o^2}. \tag{33}$$

By employing the relations $\omega = 2\pi f$, $\omega_o = 2\pi f_o$ and dividing both numerator and denominator of equation (33) by 2π, the expression for θ becomes

$$\theta = \tan^{-1} \frac{2s\,(f^3 f_o - f f_o^3)}{(f^4 - 2f^2 f_o^2 + f_o^4) - s^2 f^2 f_o^2}. \tag{34}$$

It was pointed out previously that the substitution of $f = 10^F$ results in a tangent plot of F versus θ, symmetrical about some particular value, F_n. Using the above relation for f in equation (34),

$$\theta = \tan^{-1} \frac{2s\,(10^{3F+F_o} - 10^{F+3F_o})}{(10^{4F} - 2\cdot 10^{2F+2F_o} + 10^{4F_o}) - s^2\cdot 10^{2F+2F_o}} \tag{35}$$

At $F = F_o$, $\theta = 180^\circ$, at $F = -\infty$, $\theta = 0^\circ$ and at $F = +\infty$, $\theta = 360^\circ$. Reference to the tabulated values in Appendix II discloses that for a value of $F_o = 2$, equal departures in plus and minus directions from this value result in equal angles either being added to or subtracted from 180°. It is evident from equation (35) that a change in the value of F_o

merely shifts the entire curve such that it becomes centered on a new F_o. Thus, the value F_n referred to above is, in reality, F_o. The fact that the curve may be shifted provides a convenient means of plotting curves with equal values of s but with different values of F_o. A plot of f (on a common logarithm basis) versus θ is shown in Figure 7 for various values of s.

By properly selecting f_{oA}, f_{oB} and a common value for s, the phase difference between the curves of networks A and B equals, approximately, the desired 90° over a range of frequencies. Since practically a straight line relationship between phase and frequency exists for each network over the region under consideration, a knowledge of the average slope of the curves is useful in determining f_{oA} and f_{oB}. The region over which the 90° phase shift is obtained is symmetrical about a value of F_M midway between F_{oA} and F_{oB}. This value of F is the arithmetic mean between F_{oA} and F_{oB} and, in accordance with the definition of f in terms of F, is the geometric mean between f_{oA} and f_{oB}.

The average slope at F_{oA} and F_{oB} is an indication of the trend of the curves throughout the "straight" portion thereof. The most ideal condition would be for the actual curves to coincide with an average slope line.

Since the curves of θ versus $\log_{10} f$ (or F) can never coincide exactly with any straight line, Dome[6] chose f_{oA} and f_{oB} such that the phase difference between the curves of networks A and B would be exactly 90° at, at least, f_M. Straight lines which passed through (f_{oA}, 180°),

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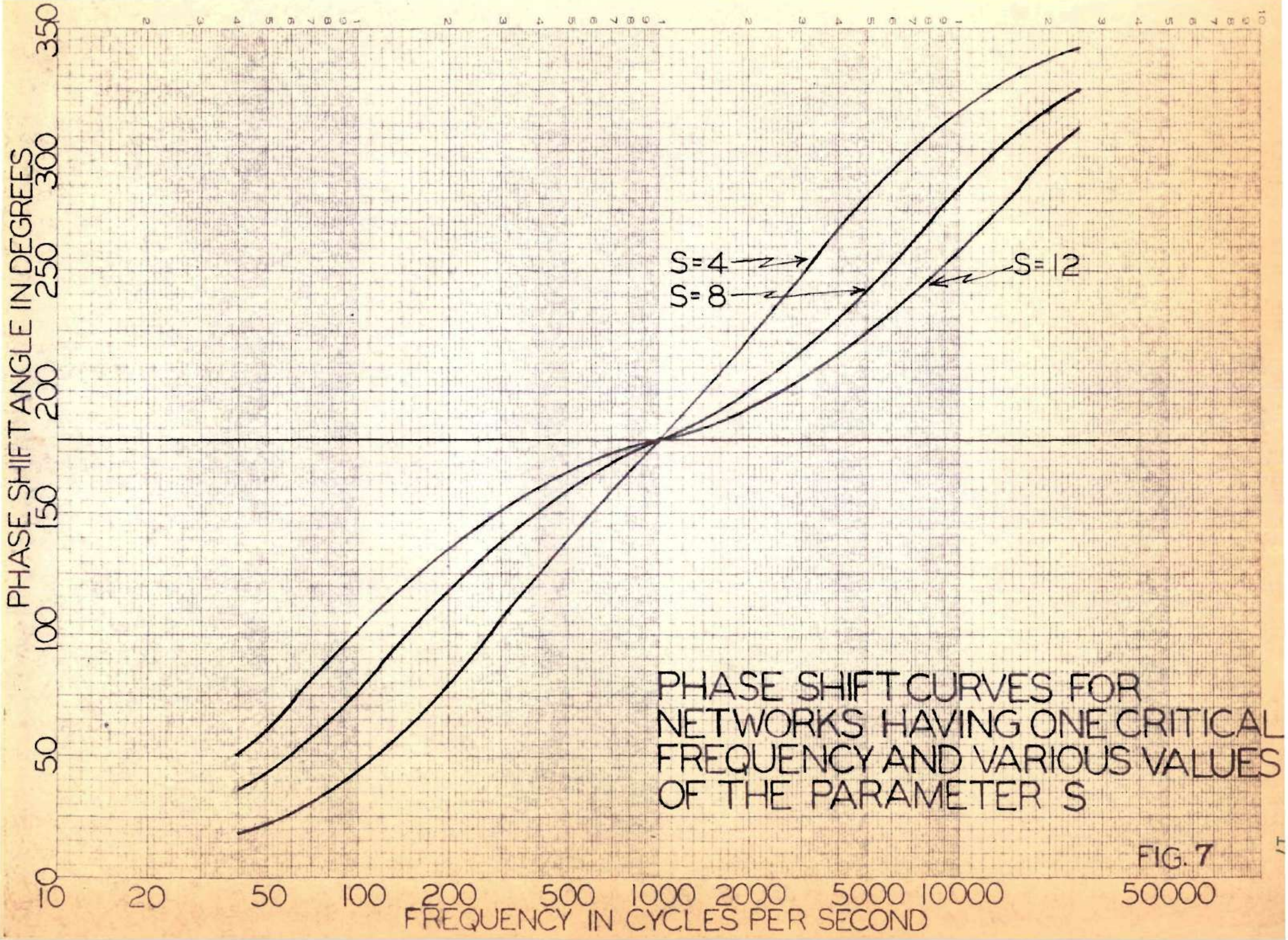
[6] Dome, R. B., loc. cit.



PHASE SHIFT ANGLE IN DEGREES
0
50
100
150
200
250
300
350
10
20
50
100
200
500
1000
2000
5000
10000
50000
FREQUENCY IN CYCLES PER SECOND
S=4
S=8
S=12
PHASE SHIFT CURVES FOR NETWORKS HAVING ONE CRITICAL FREQUENCY AND VARIOUS VALUES OF THE PARAMETER S

FIG. 7

(f_M, 225°) and f_{oB}, 180°), (f_M, 135°) would then maintain a vertical difference of 90° and could be considered as average slope lines for the curves of networks A and B. Any departures from the average slope lines would represent errors between the desired and actual conditions for each curve, which might or might not be compensated for by departures of the other curve. The difference in departures in degrees at each value of f would, therefore, be an indication of the amount the phase difference varied from 90°.

Equation (35) expresses the general relationship between θ and F. Under the conditions stated above, at F_M, θ_B must equal -135°. The general F_o in equation (35) is equal to F_{oB} for $\theta = \theta_B$. By letting g_D represent the spacing in units of F between F_M and F_{oB},

$$F_M = F_{oB} - g_D. \tag{36}$$

Then, at $F = F_M$, $\theta_B = -135°$ and $\tan\theta_B = 1$, whereby

$$1 = \frac{2s\left(10^{3F_{oB}-3g_D+F_{oB}} - 10^{F_{oB}-g_D+3F_{oB}}\right)}{\left(10^{4F_{oB}-4g_D} - 2\cdot 10^{2F_{oB}-2g_D+2F_{oB}} + 10^{4F_{oB}}\right) - s^2\cdot 10^{2F_{oB}-2g_D+2F_{oB}}}. \tag{37}$$

After factoring out 10^{4F_o} from the numerator and denominator of the right hand fraction and transposing terms, the following equation represents the relationship between g_D and s, which must be satisfied:

$$10^{-4g_D} - 2\cdot 10^{-2g_D} + 1 - s^2\cdot 10^{-2g_D} + 2s\cdot 10^{-g_D} - 2s\cdot 10^{-3g_D} = 0. \tag{38}$$

By multiplying equation (38) by 10^{4g_D} and setting s equal to 4, the value chosen by Dome, there results the relationship:

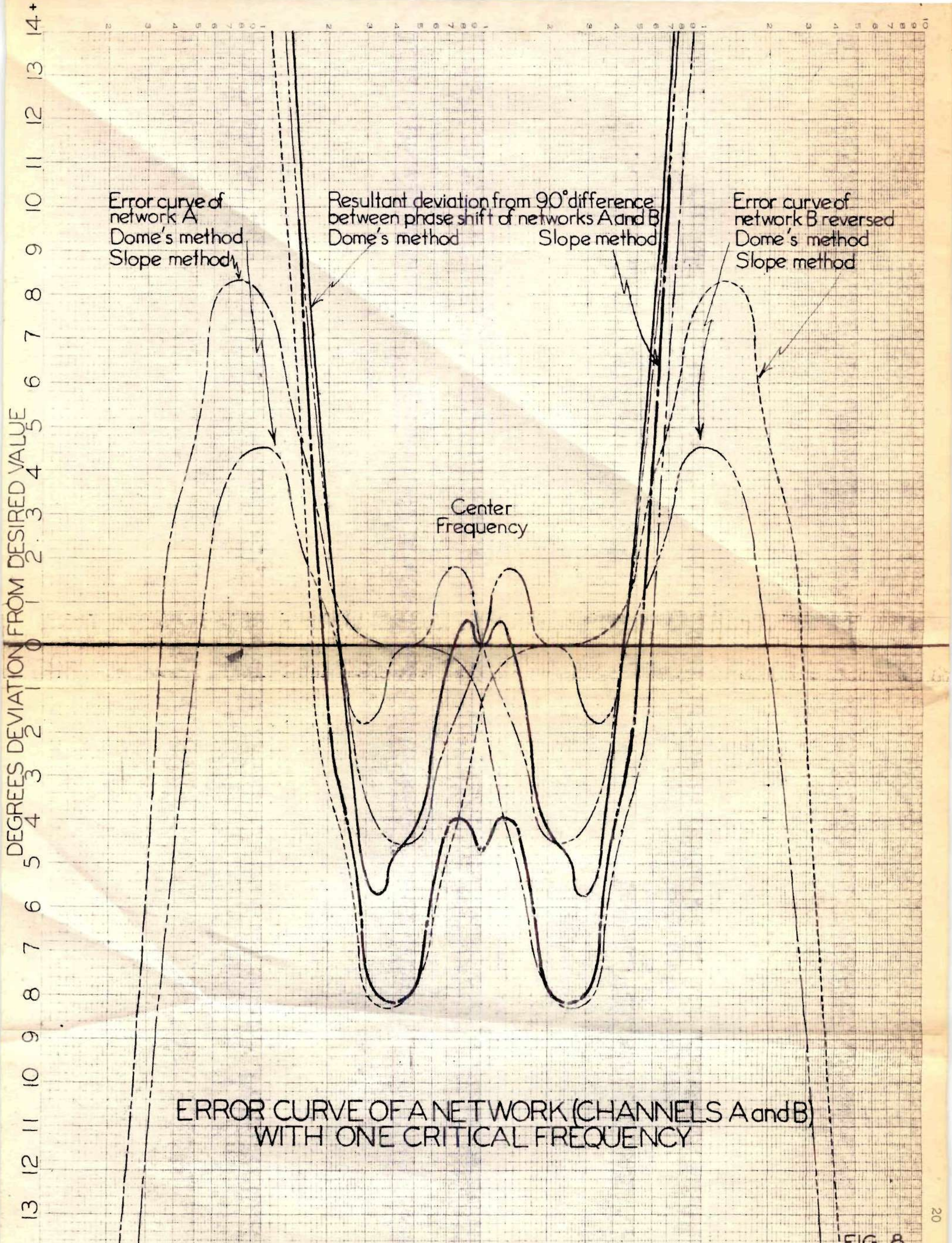
$$1-18\cdot 10^{2g_D}+10^{4g_D}+8\cdot 10^{3g_D}-8\cdot 10^{g_D}=0. \qquad (39)$$

A value of g_D equal to .328 satisfies the above equation. The slope, in units of F, of the average slope line using a value of g_D equal to .328 is 137.1° per unit of F.

Figure 8 shows a curve of the calculated error from a 90° difference for networks A and B, employing a value of s equal to 4 and a spacing between F_{oA} and F_{oB} equal to 2 x 0.328. The curve is obtained by adding the angular difference between calculated values of θ_A and the corresponding average slope line to the similar difference, reversed in sign, of network B. The maximum departure from the 90° value is slightly over 5°, which agrees substantially with Dome's calculation of 90±4° accuracy.

The question naturally arises as to whether or not Dome's method for setting f_{oA} and f_{oB} is the one which yields the best results. Further, the results of a second method should indicate, by the error obtained, the criticalness of setting f_{oA} and f_{oB}.

It follows that another logical method by which to choose f_{oA} and f_{oB} and their corresponding F's is to assume that the phase curves of networks A and B continue colinearly with the slope lines at f_{oA} and f_{oB}, and then to select a difference, $F_{oB}-F_{oA}$, such that the vertical difference between the straight slope lines is 90°. The slope lines in this, the "Slope" method, bear a correspondence to the average slope lines discussed in connection with Dome's method.



Error curve of network A
Dome's method
Slope method
Resultant deviation from 90° difference between phase shift of networks A and B
Dome's method
Slope method
Error curve of network B reversed
Dome's method
Slope method
Center Frequency
DEGREES DEVIATION FROM DESIRED VALUE
FREQUENCY IN CYCLES PER SECOND
ERROR CURVE OF A NETWORK (CHANNELS A and B) WITH ONE CRITICAL FREQUENCY

FIG. 8

Any departure of the phase curves of networks A and B from their average slope lines is, as before, a factor in determining the deviation from a true 90° difference between the phase curves, because of the symmetrical position of F_M with respect to F_{oA} and F_{oB}, once F_M is selected.

The magnitude of the distance along the $\log_{10} f$ (or F) axis from F_M to either F_{oA} or F_{oB} is given by

$$45°/m = F_M - F_{oA} \tag{40a}$$
$$45°/m = F_{oB} - F_M \tag{40b}$$

where m (expressed in degrees per unit of F) is the slope of the phase curves at either F_{oA} or F_{oB}.

The slope of the phase curves in terms of $\log_{10} f$ (or F) is determined by differentiating

$$\theta_{(\text{IN RADIANS})} = \tan^{-1} \frac{2S(f^3 f_o - f f_o^3)}{(f^4 - 2f^2 f_o^2 + f_o^4) - S^2 f^2 f_o^2} \tag{41}$$

with respect to f and then changing variables from radians to degrees and from frequency to the common logarithm of frequency.

Using the form

$$\frac{d(\tan^{-1} v)}{du} = \frac{1}{1 + v^2} \frac{dv}{du}, \tag{42}$$

$d\theta/df$ is

$$\frac{d\theta}{df}=\frac{1}{1+\left[\frac{2s(f^3f_0-ff_0^3)}{(f^4-2f^2f_0^2+f_0^4)-s^2f^2f_0^2}\right]^2}\cdot\frac{d}{df}\left[\frac{2s(f^3f_0-ff_0^3)}{(f^4-2f^2f_0^2+f_0^4)-s^2f^2f_0^2}\right] \quad (43)$$

which reduces to

$$\frac{d\theta}{df}=\frac{f^6f_0(-2s)+f^4f_0^3(-2s^3+2s)+f^2f_0^5(-2s^3+2s)+f_0^7(-2s)}{f^8+f^6f_0^2(2s^2-4)+f^4f_0^4(s^4-4s^2+6)+f^2f_0^6(2s^2-4)+f_0^8} \quad (44)$$

measured in radians per cycle per second.[7] Since

$$\frac{d\theta\,(\text{RADIANS})}{d(\log_{10}f)}=\frac{d\theta}{dF}=\frac{d\theta}{df}\cdot\frac{df}{dF} \quad (45)$$

and

$$\theta_{(\text{DEGREES})}=\frac{180}{\pi}\cdot\theta\,(\text{RADIANS}) \quad (46)$$

and

$$\frac{df}{dF}=\frac{d(10^F)}{dF}=10^F\log_\epsilon 10 \quad (46a)$$

then

$$\frac{d\theta}{dF}=(\log_\epsilon 10)\frac{180}{\pi}\left[\frac{10^{7F+F_0}\cdot(-2s)+10^{5F+3F_0}\cdot(-2s^3+2s)+10^{3F+5F_0}\cdot(-2s^3+2s)+10^{F+7F_0}\cdot(-2s)}{10^{8F}+10^{6F+2F_0}\cdot(2s^2-4)+10^{4F+4F_0}\cdot(s^4-4s^2+6)+10^{2F+6F_0}\cdot(2s^2-4)+10^{8F_0}}\right] \quad (47)$$

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[7] Hudson, R. G., The Engineer's Manual (New York, John Wiley & Sons, Inc., 1939) p. 32.

measured in degrees per unit of F.

At $F = F_o$

$$m = \frac{d\theta}{dF} = -\frac{720}{\pi}(\log_\epsilon 10). \tag{47a}$$

For a value of s equal to 4, the slope m of the phase curves at f_{oA} and f_{oB} (or the corresponding F's) is equal to 132^o per unit of F. Substitution of the above value of m in equations (40a) and (40b) yields

$$F_{OA} = F_M - 45/132 \;;\; F_{OA} = F_M - .341 \tag{48a}$$
$$F_{OB} = F_M + 45/132 \;;\; F_{OB} = F_M + .341. \tag{48b}$$

In accordance with the previous convention of associating g_D with the difference between F_M and either F_{oA} or F_{oB} in Dome's method, the above value of .341 is designated g_S to indicate that it is the corresponding spacing in units of F determined by the Slope method.

A plot of calculated deviation from a 90^o difference for s equal to 4 and g_S equal to .341 is shown in Figure 8. The method of obtaining this curve is identical to that used in obtaining the error curve of Dome's method. The maximum deviation from a 90^o phase difference is approximately $\pm 8^o$.

Comparison of the difference from a value of 90^o obtained by Dome's method with that obtained by the Slope method indicates that Dome's method is the better. The difference in maximum error demonstrates that the choice of a value of g is critical. The overall curves are similar except for a general downward shift of the Slope method curve.

An inspection of the two curves suggests that the value of g chosen by Dome is not necessarily the optimum which can be obtained.

DETERMINATION OF PHASE-SHIFT CONSTANTS FOR A TYPICAL ONE POLE L-C SYSTEM

To derive the circuit parameters of two single pole L-C networks, each of which will have phase characteristics identical to those shown in Figure 7 for $s = 4$, the following procedure applies, using Dome's method:

(1) Select a value for f_M and determine $F_M = \log_{10} f_M$.

(2) Add .328 to F_M to obtain F_{OB}; subtract .328 from F_M to obtain F_{OA}.

(3) Determine f_{oA} and f_{oB} from relationship $10^F = f$.

For Channel A:

(4) Select a suitable value of C_{1A}.

(5) C_{2A} then equals $s^2 C_{1A}$ or $16C_{1A}$.

(6) Determine L_{1A} and L_{2A} such that $L_{1A} = 1/(2\pi f_{oA})^2 C_{1A}$; $L_{2A} = 1/(2\pi f_{oA})^2 C_{2A}$.

(7) Determine R_{3A} such that $R_{3A} = \frac{1}{2} L_{1A}/C_{2A}$.

For Channel B:

(8) Select a suitable value of C_{1B}.

(9) C_{2B} then equals $s^2 C_{1B}$ or $16C_{1B}$.

(10) Determine L_{1B} and L_{2B} such that $L_{1B} = 1/(2\pi f_{oB})^2 C_{1B}$; $L_{2B} = 1/(2\pi f_{oB})^2 C_{2B}$.

(11) Determine R_{3B} such that $R_{3B} = \frac{1}{2} L_{1B}/C_{2B}$.

The order of determining the circuit constants is immaterial. In certain instances, it might be more feasible to select the inductances first and then determine the capacitances.

The above derivation of circuit parameters is based on the assumption that the reactive elements of the circuit are perfect inductances and capacitances. However, it is not possible to obtain inductances and capacitances which do not have equivalent resistive components in some degree. The difference between ideal and actual conditions is, therefore, governed by the factor of merit, or Q, of the reactive elements. Magnetic coupling between inductances also is a contributing factor to the departure of the networks from ideal conditions.

Solutions of networks containing pure reactances are obtained more easily, in general, than solutions of networks having mixed resistances and reactances. Equation (34) is a solution of the one pole L-C network for θ in terms of f and s. The significant fact about this equation is that θ is completely defined by an expression involving only frequencies and a constant, s. If the same expression for θ in terms of f and a constant, s, is obtained for a network configuration, that network in itself has the same phase characteristics as the L-C network.

It will be shown later that various R-C combinations, which are more practical than L-C combinations, are defined for θ in terms of f and s in accordance with equation (34). The pattern of solution for more complex R-C networks is determined from the solution of L-C networks containing more parameters than the one previously discussed. In the latter solutions, more sophisticated methods of circuit analysis are employed. The basis for these methods will be discussed at this point.

METHODS OF NETWORK REPRESENTATION AND REDUCTION

According to Foster[8], the impedance of a network containing only reactive elements is represented by the general formula

$$Z = kp\frac{(p^2-p_2^2)(p^2-p_4^2)------(p^2-p_m^2)}{(p^2-p_1^2)(p^2-p_3^2)------(p^2-p_{m-1}^2)} \tag{49}$$

where $p = j\omega = j2\pi f$, the p_m^2 's are negative real quantities equal to $-\omega_m^2$ and k is a constant. By imposing the condition that

$$-p_m^2 \geqslant -p_{m-1}^2 \geqslant ------ \geqslant -p_2^2 \geqslant -p_1^2 \geqslant 0 \tag{50}$$

the zeros and poles are made to alternate along the frequency axis. The p_m's are those values of p at which the impedance Z may become infinite or zero. As p approaches a p_m in the numerator, the impedance approaches zero. The impedance is said to have a zero at the value of frequency corresponding to the particular p_m. Similarly, as p approaches a p_m in the denominator, the impedance expression becomes infinite. A pole of the impedance, therefore, exists a p equal to p_m.

Specifically, four general types of reactive networks may be represented by formula (49). These are the L-L type which has the characteristics of an inductance at both zero and an infinite frequency, the L-C type which has the characteristics of an inductance at zero

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[8]Foster, R. M., "A Reactance Theorem", Bell System Technical Journal, April, 1924, pp. 259-267.

frequency and of a capacitance at an infinite frequency, the C-L type which has the characteristics of a capacitance at zero frequency and an inductance at an infinite frequency and the C-C type which behaves like a capacitance at both zero and an infinite frequency. Depending upon the number of elements in any given type of network, the impedance while remaining reactive in character, may assume zero and infinite values many times as the frequency is varied from zero to infinity. The general formulae for the four cases are given below.

$$\text{L-L} \qquad Z = pL\,\frac{(p^2-p_2^2)(p^2-p_4^2)-----(p^2-p_m^2)}{(p^2-p_1^2)(p^2-p_3^2)------(p^2-p_{m-1}^2)} \tag{51}$$

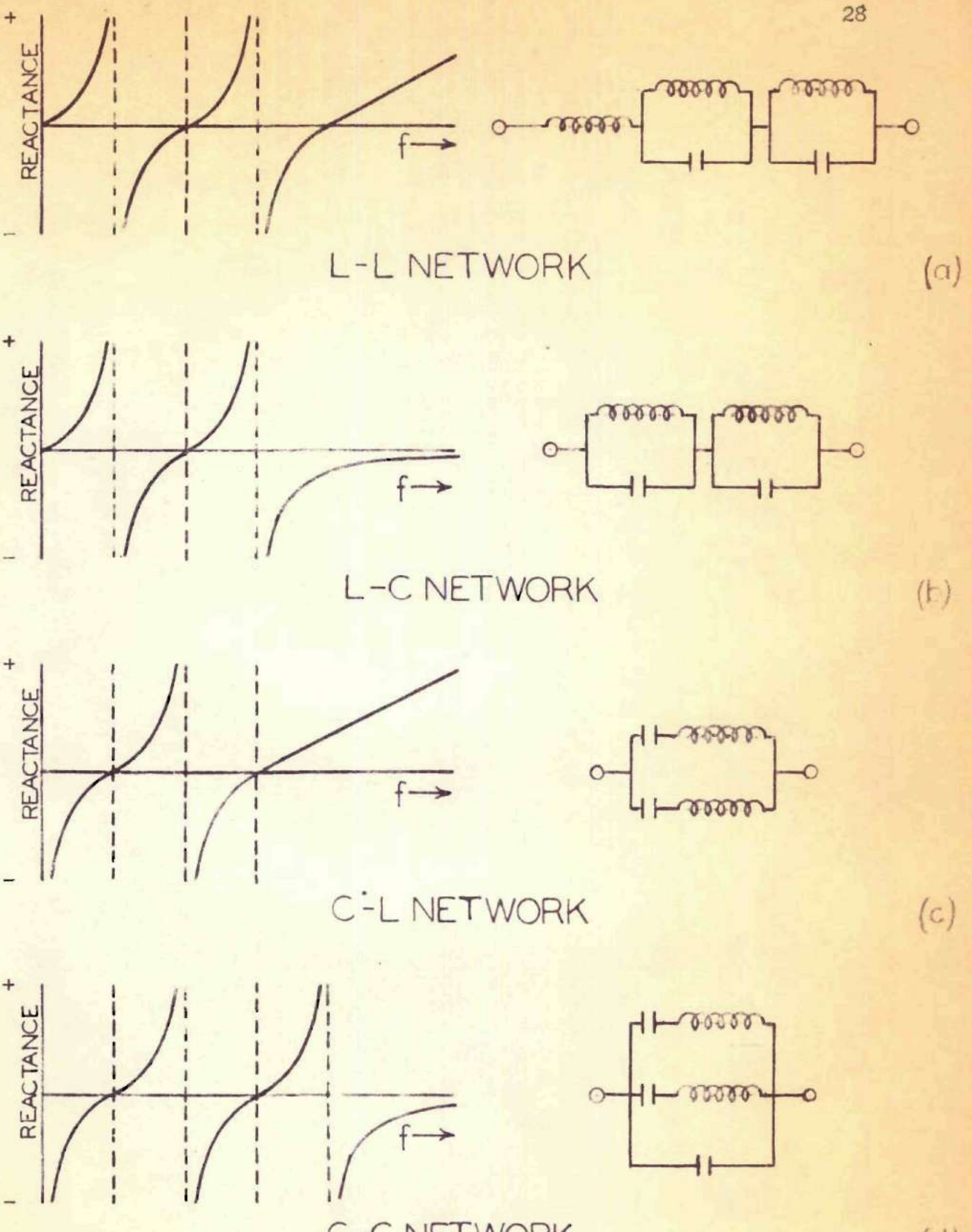
$$\text{L-C} \qquad Z = \frac{p}{C}\,\frac{(p^2-p_2^2)(p^2-p_4^2)----(p^2-p_{m-1}^2)}{(p^2-p_1^2)(p^2-p_3^2)---(p^2-p_m^2)} \tag{52}$$

$$\text{C-L} \qquad Z = \frac{L}{p}\,\frac{(p^2-p_1^2)(p^2-p_3^2)----(p^2-p_m^2)}{(p^2-p_2^2)(p^2-p_4^2)----(p^2-p_{m-1}^2)} \tag{53}$$

$$\text{C-C} \qquad Z = \frac{1}{pC}\,\frac{(p^2-p_1^2)(p^2-p_3^2)---(p^2-p_{m-1}^2)}{(p^2-p_2^2)(p^2-p_4^2)---(p^2-p_m^2).} \tag{54}$$

The impedance of the constant term is the impedance of the circuit at an infinite frequency. The simple parallel and series L-C circuits, which constitute the impedance arms of the phase-shift network previously discussed, are special cases of formulae (52) and (53), respectively. Figure 9 shows the characteristics of the various types of networks, together with the network configurations to produce the characteristics.

In later applications, it will be desirable to determine the individual circuit constants of a network, knowing the equivalent



CHARACTERISTICS OF REACTIVE NETWORKS

FIG. 9

L or C at an infinite frequency and knowing the location of the zeros and poles. Bode[9] has developed methods of network reduction by which an impedance Z, expressed as a function of p, is represented in terms of R, L or C. The derived network may not necessarily be the exact configuration of the unknown Z, but its frequency-impedance characteristics will exactly duplicate those of the unknown Z. In Bode's method, the impedance or admittance (the reciprocal of impedance) of a network arm is evaluated, and that impedance (or admittance) is then subtracted from the original impedance (or admittance) to obtain a new impedance (or admittance) on which the process is repeated until all element values have been obtained. The formulae (51) through (54) are particularly adaptable to network reduction, since the network configuration is known and since no resistive terms are present. Knowing the equivalent value and type of the circuit element which predominates at an infinite frequency and knowing the location of poles and zeros (as specified by the p_m^2 's), it is a simple matter to apply Bode's methods and evaluate the circuit parameters. The expression for Z may contain, in addition to the p_m^2's, a single p which represents a single pole or zero. The p_m^2's represent double poles or zeros.

The type of circuit branches which may be extracted from pure reactive networks separately are parallel L-C combinations, a single C, series L-C combinations and a single L. Inspection of the networks

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[9]Bode, H. W., Network Analysis and Feedback Amplifier Design (New York, D. Van Nostrand Co., Inc., 1946) pp. 170-195.

of Figure 9 bears this out for specific cases. Bode[10] has demonstrated that as p approaches either of the double poles, represented by p_ℓ^2, the expression for Z approaches

$$Z = \frac{2A_o p}{(p^2 - p_\ell^2)} \qquad (55)$$

which may be compared to the impedance of a parallel L-C combination,

$$Z_{L-C\,(PARALLEL)} = \frac{p/C}{(p^2 + 1/LC)}. \qquad (56)$$

By imposing the conditions

$$1/C = 2A_o \ ; \quad L = -\frac{2A_o}{p_\ell^2} \qquad (57)$$

and evaluating A_o by the relation

$$A_o = \lim_{p \to p_\ell} \left[\frac{(p^2 - p_\ell^2)}{2p_\ell}\right] Z, \qquad (58)$$

the value of the constants of the parallel L-C combination may be determined. The impedance of the parallel L-C combination may then be subtracted from the total impedance and the process repeated for the new impedance to remove the next parallel L-C combination, if one exists.

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[10]Bode, H. W., loc. cit.

In addition to double poles, the Z of formula (51) contains a single pole of impedance at p equal to zero if p is interpreted as equal to (p-0). A single capacitance also has a pole of impedance at p equal to zero. The corresponding representation for the impedance of Z as p approaches zero is

$$Z = \frac{A_o}{(p - 0)}. \tag{59}$$

The impedance of a single capacitance is

$$Z_c = 1/pC = \frac{1}{C(p-0)}. \tag{60}$$

The value of A_o in this case must be $1/C = A_o$ for an equivalence between formulae (59) and (60). A_o may be evaluated as

$$A_o = \lim_{p \to 0} pZ. \tag{61}$$

Thus, if any single capacitance exists, it may be readily removed.

The two remaining types of circuit branches to be considered are the series L-C combination and the single inductance, L. The concept of admittance reduction may be applied to these branches. Admittance is defined as the reciprocal of impedance. Thus, a zero of impedance becomes a pole of admittance. As p approaches either of the double poles of admittance represented by p_n^2, the admittance, Y, approaches

$$Y = \frac{2A_0 p}{(p^2 - p_n^2).} \qquad (62)$$

The admittance of a series L-C combination is

$$Y_{L-C(SERIES)} = \frac{p/L}{(p^2 + 1/LC).} \qquad (63)$$

The conditions

$$1/L = 2A_0 \quad ; \quad C = -\frac{2A_0}{p_n^2} \qquad (64)$$

where

$$A_0 = \lim_{p \to p_n} \left[\frac{(p^2 - p_n^2)}{2p_n^2} \right] \qquad (65)$$

equate the Y of expression (62) to the Y of the series L-C branch. Series L-C branches may be extracted and the process repeated on the remaining admittance to extract any additional series L-C branches.

Single poles of admittance at p equal to zero may exist. In this case, the admittance approaches

$$Y = \frac{A_0}{(p - 0)} \qquad (66)$$

as p approaches zero. The admittance of a single inductance is

$$Y_L = \frac{1}{pL} = \frac{1}{L}\frac{1}{(p-0)}. \qquad (67)$$

By making 1/L equal to A_o, the correspondence between equations (66) and (67) is established. A_o may be determined by

$$A_o = \lim_{p \to 0} (p-0)Y. \qquad (68)$$

The admittance of the single inductance may then be removed.

If the value of the circuit constant at an infinite frequency and the location of poles and zeros are known, the circuit parameters of any of the impedance expressions, formulae (51) through (54), may be determined by the methods outlined above. The previous knowledge of the network configuration is helpful but not necessary.

ANALYSIS OF SINGLE POLE L-C NETWORK BY POLES AND ZEROS

As a guide to later extensions of the L-C type of network, the single pole L-C network is briefly analyzed by methods discussed in the previous section on network reduction.

In accordance with Foster's reactance representation, the impedance of the parallel L-C combination of Figure 4 is

$$Z_1 = \frac{p}{C_1}\frac{1}{(p^2 - p_o^2)} \qquad (69)$$

where

$$p_0^2 = -1/L_1C_1. \quad (70)$$

C_1 is obviously the equivalent capacitance of the branch at an infinite frequency.

Likewise, the impedance of the series L-C combination is

$$Z_2 = \frac{L_2}{p}\frac{(p^2 - p_0^2)}{1} \quad (71)$$

where

$$p_0^2 = -1/L_2C_2. \quad (72)$$

L_2 is the equivalent inductance of the branch at an infinite frequency.

It was previously shown in equation (30) that

$$s^2 = L_1/L_2\ ;\ L_2 = L_1/s^2. \quad (73)$$

The significance which may be attached to this fact is that s^2 is the ratio of the equivalent inductance of Z_1 at zero frequencies to the equivalent inductance of Z_2 at an infinite frequency.

By substituting the value of L_2 from equation (73) in equation (71), Z_2 becomes

$$Z_2 = L_1/s^2p\,(p^2 - p_0^2). \quad (74)$$

The impedance Z_3 is the resistance, R_3, of Figure 4. Accordingly,

$$Z_3 = R_3 . \tag{75}$$

It was previously shown in equation (15) that

$$4R_3^2 = L_1/C_2 = L_2/C_1 . \tag{76}$$

From the fact that $L_1/L_2 = C_2/C_1$, equation (73) indicates that

$$C_2 = C_1 S^2 . \tag{77}$$

After substituting the value of C_2 from equation (77), transposing terms and taking the square root of each side, equations (75) and (76) reduce to

$$Z_3 = R_3 = \frac{1}{2S}\sqrt{L_1/C_1} . \tag{78}$$

The ratio

$$\frac{(Z_1 - Z_2)}{\frac{Z_1 Z_2}{Z_3} + (Z_1 + Z_2)} \tag{79}$$

when evaluated becomes

$$\frac{p^4/p_0^2 - 2p^2 + p^2S^2 + p_0^2}{2\sqrt{L_1C_1}\,S p^3 - 2\sqrt{L_1C_1}\,S p p_0^2 + p^2S^2 - p^4/p_0^2 + 2p^2 - p_0^2}. \quad (80)$$

Substitution of $j\omega$ for p and $1/\omega_0$ for L_1C_1 further reduces expression (80) to

$$\frac{-\omega^4/\omega_0^2 - 2\omega^2 - \omega^2 s^2 - \omega_0^2}{\omega^4/\omega_0^2 - 2\omega^2 - \omega^2 s^2 + \omega_0^2 - j2s\,\omega^3/\omega_0 + j2s\,\omega\omega_0}. \quad (81)$$

The phase angle of expression (81) is obtained by evaluating the negative phase angle of the reciprocal of expression (81), which yields

$$\theta = \tan^{-1} \frac{2s(\omega^3\omega_0 - \omega\omega_0^3)}{(\omega^4 - 2\omega^2\omega_0^2 + \omega_0^4) - s^2\omega^2\omega_0^2} \quad (82)$$

and its counterpart

$$\theta = \tan^{-1} \frac{2s(f^3 f_0 - f f_0^3)}{(f^4 - 2f^2 f_0^2 + f_0^4) - s^2 f^2 f_0^2}. \quad (83)$$

The expression for θ is the same as was obtained by the previous analysis of the single pole L-C network. The relationships between the circuit constants are also the same as before.

ANALYSIS OF R-C NETWORK WITH ISOLATING VACUUM TUBE

It will now be demonstrated that the network shown in Figure 5 possesses the same phase characteristics as the network of Figure 4. Two vacuum tubes, however, are required as compared to the single tube needed in the network of Figure 4 and, as will be shown later, in the network of Figure 6. It is readily seen that the network in its entirety is composed of two meshes, of the type considered in Appendix I, coupled through a vacuum tube circuit. The vacuum tube through which the two meshes are coupled serves the dual purpose of isolating one mesh from the other and providing a means of obtaining the two, equal voltages of opposite polarity, which are used as the voltage sources for the second mesh. For this network, it is assumed that constant attenuation without appreciable phase shift occurs in the coupling vacuum tube circuit. Under these conditions, the input voltages to the second mesh are each a set fraction of the output voltage of the first mesh and are either in phase or out of phase with the first mesh output, depending upon which voltage is considered. The general mesh equation, which is developed in Appendix I, is

$$V_{34} = V_{13}\frac{(Z_1 - Z_2)}{\frac{Z_1 Z_2}{Z_3} + (Z_1 + Z_2)} \,. \tag{84}$$

As applied to the first mesh, equation (84) becomes

$$V'_{34} = V'_{13}\frac{(-1/pC'_1 - R'_1)}{(-\frac{1}{pC'_1} + R'_1)} = V_{13}\frac{(-1 - j\omega R'_1 C'_1)}{(1 - j\omega R_1 C'_1)} \tag{85}$$

wherein R_1' is Z_1, $1/pC_1'$ is Z_2 and Z_3 is infinite. The impedance Z_3 in the first mesh is the input impedance of the coupling vacuum tube, which is very high compared to the other circuit impedances and for all practical purposes may be considered infinite.

The voltages V_{13} and V_{23} which appear in the second mesh are thus equal to

$$K'V_{13}'\frac{(-1 - j\omega R_1'C_1')}{(+1 - j\omega R_1'C_1')} \tag{86}$$

and

$$-K'V_{13}'\frac{(-1 - j\omega R_1'C_1')}{(1 - j\omega R_1'C_1')}, \tag{87}$$

respectively, K' being a fixed constant of the vacuum tube coupling. Accordingly, the voltage V_{34} which appears at the output terminals of the network is equal to

$$V_{34} = \left[\frac{K'V_{13}'(-1 - j\omega R_1'C_1')}{(1 - j\omega R_1'C_1')}\right]\left[\frac{(-1 - j\omega R_1C_1)}{(1 - j\omega R_1C_1}\right] \tag{88}$$

where the first bracketed expression is the reference driving voltage, R_1 is the Z_1 and $1/pC_1$ is the Z_2 of the second mesh. Again, the impedance corresponding to Z_3 must be theoretically infinite or in a practical application, very high. This is accomplished by applying the output of the network to a vacuum tube amplifier stage having a high input-to-output impedance ratio.

By recombining the terms of equation (88) as follows

$$V_{34} = KV_{13}'\left[\frac{(1-\omega^2 R_1'C_1'R_1C_1) + j(\omega R_1'C_1' + \omega R_1C_1)}{(1-\omega^2 R_1'C_1'R_1C_1) - j(\omega R_1'C_1' + \omega R_1C_1)}\right], \tag{89}$$

it is seen that for all values of ω, the bracketed expression is the quotient of a complex number and its conjugate. The magnitude of such an expression is equal to unity. The amplitude of the output voltage thus remains a constant, fixed fraction of the input voltage of the entire network.

The phase angle is determined by rationalizing equation (89) and finding the arctangent of the quotient formed by the imaginary component divided by the real component. Thus,

$$V_{34} = K'V_{13}'\left[\frac{(1-\omega^2 R_1'C_1'R_1C_1)^2 - (\omega R_1'C_1' + \omega R_1C_1)^2}{(1-\omega^2 R_1'C_1'R_1C_1)^2 + (\omega R_1'C_1' + \omega R_1C_1)^2} + j\,\frac{2(\omega R_1'C_1' + \omega R_1C_1)(1-\omega^2 R_1'C_1'R_1C_1)}{(1-\omega^2 R_1'C_1'R_1C_1)^2 + (\omega R_1'C_1' + \omega R_1C_1)^2}\right] \tag{90}$$

and

$$\theta = \tan^{-1} 2\,\frac{(R_1'C_1' + R_1C_1)(-\omega^3 R_1'C_1'R_1C_1 + \omega)}{(\omega^4 R_1'^2C_1'^2R_1^2C_1^2 - 2\omega^2 R_1'C_1'R_1C_1 + 1) - \omega^2(R_1'C_1' + R_1C_1)^2}. \tag{91}$$

By letting

$$\omega_0^2 = \frac{1}{R_1'C_1'R_1C_1}, \tag{92}$$

$$\Theta = \tan^{-1} \frac{2\left[\frac{R_1'C_1' + R_1C_1}{\sqrt{R_1'C_1'R_1C_1}}\right]\left(\omega\omega_0^3 - \omega^3\omega_0\right)}{\left[\omega^4 - 2\omega^2\omega_0^2 + \omega_0^4\right] - \omega^2\omega_0^2\left[\frac{(R_1'C_1' + R_1C_1)^2}{R_1'C_1'R_1C_1}\right]}. \tag{93}$$

In order for equation (93) to be of the form of equation (33), the following condition must be imposed on the circuit constants:

$$\frac{R_1'C_1' + R_1C_1}{\sqrt{R_1'C_1'R_1C_1}} = s. \tag{94}$$

It is evident that in general $R_1'C_1'$ should not equal R_1C_1, since the value of s would thereby be restricted to 2. However, by letting $R_1' = R_1$ and $C_1' = a'C_1$, where a' is a constant, s becomes

$$s = \frac{a'R_1C_1 + R_1C_1}{\sqrt{a'R_1^2C_1^2}} = \frac{a'+1}{\sqrt{a'}} \tag{95}$$

or s^2 becomes

$$s^2 = \frac{(a'^2 + 2a' + 1)}{a'} \tag{96}$$

from which

$$a' = \frac{s^2 - 2 \pm \sqrt{s^4 - 4s^2}}{2}. \qquad (97)$$

To avoid imaginary components in the above expression for a', s must be equal to or greater than 2.

The value of f_o and its attendant F_o is determined from

$$f = \omega/2\pi \; ; \; f_o = \omega_o/2\pi \qquad (98)$$

$$f_o = \frac{1}{2\pi}\frac{1}{R_1C_1\sqrt{a'}} = \frac{1}{2\pi}\frac{1}{\sqrt{R_1'C_1'R_1C_1}} \qquad (99)$$

and

$$F_o = \log_{10}\frac{1}{2\pi}\frac{1}{\sqrt{R_1'C_1'R_1C_1}}. \qquad (100)$$

With a predetermined value of s and the required relationship between f_o (or F_o) and the circuit elements, the expression for θ reduces to

$$\theta = \tan^{-1}\frac{2s\,(f^3f_o - f f_o^3)}{(f^4 - 2f^2f_o^2 + f_o^4) - s^2 f^2 f_o^2} \qquad (101)$$

and

$$\theta = \tan^{-1} \frac{2s(10^{3F+F_0} - 10^{F+3F_0})}{(10^{4F} - 2\cdot 10^{2F+2F_0} + 10^{4F_0}) - s^2 \cdot 10^{2F+2F_0}} \quad (102)$$

which are the same expressions as were previously developed for the circuit of Figure 4.

A parallel procedure to that previously used for the one pole L-C network should be followed in determining the values of f_{oA}, f_{oB}, etc., for channels A and B employing networks of the type above described.

Networks of this type are more practical than the L-C networks, due to the difficulty of obtaining inductances of sufficiently high Q and of preventing interaction between inductances. However, the following analysis of Dome's third network will demonstrate that identical results are obtained by using one vacuum tube in place of two.

ANALYSIS OF R-C NETWORK WITH ONE CRITICAL FREQUENCY

The third type of network proposed by Dome[11] and shown in Figure 6 appears to be more practical than either of the two previously discussed. Not only are resistance and capacitance elements used throughout, but the required phase shift takes place in one phase-shift section. An isolating vacuum tube is not required. The appearance of the phase-shift section is somewhat similar to that of the one pole L-C network. However, series and parallel R-C combinations make up the impedance arms.

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[11]Dome, R. B., loc. cit.

The use of the terms poles and zeros is avoided, and in their stead the concept of a "critical frequency" is introduced. A series R-C combination has no zero of impedance at any frequency and has a pole of impedance only at zero frequency. A parallel combination has no pole of impedance at any frequency and has a zero of impedance only at an infinite frequency. There are frequencies between zero and infinity, however, at which the voltage drops across the condenser and resistor of a series R-C combination are equal and 90° out of phase. Similarly, there are frequencies between zero and infinity at which the currents through the condenser and resistor of a parallel R-C combination are equal and 90° out of phase. These frequencies are termed "critical frequencies". For either the series or parallel case,

$$f_c = \frac{1}{2\pi RC} \quad ; \quad \omega_c = \frac{1}{RC}, \qquad (103)$$

where f_c is the critical frequency.

From Appendix I,

$$V_{34} = V_{13} \frac{(Z_1 - Z_2)}{\frac{Z_1 Z_2}{Z_3} + (Z_1 + Z_2)} . \qquad (104)$$

The values of Z_1, Z_2 and Z_3 for the R-C network with one critical frequency are, from calculation and reference to Figure 6,

$$Z_1 = \frac{R_1}{P} \; \frac{(p + 1/R_1C_1)}{1} \qquad (105)$$

$$Z_2 = \frac{1}{C_2} \; \frac{1}{(p + 1/R_2C_2)} \qquad (106)$$

$$Z_3 = \frac{1}{C_3} \; \frac{1}{(p + 1/R_3C_3)}. \qquad (107)$$

One of the requirements of the output voltage is that its magnitude remain a constant fraction of V_{13}, which has been chosen as a reference. Although in the L-C network, the fraction was equal to unity, the most general case which can be considered contains a fractional multiplier, of which unity is a special instance. If M is the fractional multiplier, then, from equation (104), M is the magnitude of

$$\frac{(Z_1 - Z_2)}{\frac{Z_1 Z_2}{Z_3} + (Z_1 + Z_2)}. \qquad (108)$$

After substituting the values of equations (105) through (107) in the right hand term of equation (108) and setting $p = j\omega$, M becomes the magnitude of

$$\left[\frac{(1 - \omega^2 R_1 C_1 R_2 C_2) + j\omega (R_1C_1 + R_2C_2 - R_2C_1)}{(1 + R_2/R_3 - \omega^2 R_1C_1R_2C_2 - \omega^2 R_2/R_3 \cdot R_1C_1R_3C_3) + j\omega (R_2/R_3 \cdot R_1C_1 + R_2/R_3 \cdot R_3C_3 + R_1C_1 + R_2C_2 + R_2C_1)}\right]. \qquad (109)$$

If, for simplicity, $R_1C_1 = d'$, $R_2C_2 = b'$, $R_3C_3 = g'$, $R_2C_1 = h$ and $R_2/R_3 = K''$, equation (109), upon clearing fractions, becomes

$$(M + MK'' - \omega^2 M d'b' - \omega^2 MK''d'g') + j\omega(MK''d' + MK''g' + Md' + Mb' + Mh) = (1 - \omega^2 d'b') + j\omega(d' + b' - h). \quad (110)$$

By setting reals equal to reals and imaginaries equal to imaginaries, the following two equations are found:

$$M + MK'' - \omega^2 M d'b' - \omega^2 MK''d'g' = 1 - \omega^2 d'b' \quad (111)$$

$$MK''d' + MK''g' + Md' + Mb' + Mh = d' + b' - h. \quad (112)$$

To be independent of frequency, all terms containing ω must equal zero. Therefore, in equation (111), the following relationship must be satisfied:

$$\omega^2 MK''d'g' + \omega^2 M d'b' = \omega^2 d'b'. \quad (113)$$

Then, also from equation (111),

$$M + K'' = 1. \quad (114)$$

Equations (113) and (114) are satisfied if $g' = b'$.

By setting the critical frequencies of Z_1, Z_2 and Z_3 equal to $\omega_0/2\pi$,

$$1/\omega_0 = R_1C_1 = R_2C_2 = R_3C_3 \tag{115}$$

which automatically sets g' equal to b'. M is

$$M = \frac{R_3}{R_2 + R_3} \tag{116}$$

which establishes the proper ratio between the magnitudes of the input and output voltages.

The term "Z_1Z_2/Z_3" of the denominator of equation (108) reduces to $(Z_1) \cdot (R_2/R_3)$. Equation (104) may be successively simplified as follows:

$$V_{34} = V_{13}\frac{(Z_1 - Z_2)}{Z_1(R_2/R_3 + 1) + Z_2} = \left(\frac{R_3}{R_2 + R_3}\right)\frac{V_{13}(Z_1 - Z_2)}{Z_1 + Z_2\left(\frac{R_3}{R_3 + R_2}\right)} =$$

$$MV_{13}\frac{Z_1 - Z_2}{Z_1 + MZ_2} \tag{117}$$

where

$$M = \frac{R_3}{R_2 + R_3} = \frac{C_2}{C_2 + C_3}. \tag{117a}$$

Substituting the values for Z_1 and Z_2 in equation (117) results in

$$V_{34} = MV_{13}\left[\frac{(R_1 - j/\omega C_1) - \left(\dfrac{R_2 - j\omega R_2^2 C_2}{\omega^2 R_2^2 C_2 + 1}\right)}{(R_1 - j/\omega C_1) + M\cdot\left(\dfrac{R_2 - j\omega R_2^2 C_2}{\omega^2 R_2^2 C_2 + 1}\right)}\right] \tag{118}$$

which expands as

$$V_{34} = MV_{13}\left\{\begin{array}{l}\left[\dfrac{\left(R_1 - \dfrac{R_2}{\omega^2/\omega_0^2 + 1}\right)\left(R_1 + \dfrac{MR_2}{\omega^2/\omega_0^2 + 1}\right) - \left(\dfrac{R_2\,\omega/\omega_0}{\omega^2/\omega_0^2 + 1} - \dfrac{R_1}{\omega/\omega_0}\right)\left(\dfrac{MR_2\,\omega/\omega_0}{\omega^2/\omega_0^2 + 1} + \dfrac{R_1}{\omega/\omega_0}\right)}{(\text{Denominator})}\right] \\ + j\left[\dfrac{\left(R_1 + \dfrac{MR_2}{\omega^2/\omega_0^2 + 1}\right)\left(\dfrac{R_2\,\omega/\omega_0}{\omega^2/\omega_0^2 + 1} - \dfrac{R_1}{\omega/\omega_0}\right) + \left(R_1 - \dfrac{R_2}{\omega^2/\omega_0^2 + 1}\right)\left(\dfrac{MR_2\,\omega/\omega_0}{\omega^2/\omega_0^2 + 1} + \dfrac{R_1}{\omega/\omega_0}\right)}{(\text{Denominator})}\right]\end{array}\right\}. \tag{119}$$

In determining the phase angle of V_{34}, it is not necessary to consider the common denominator, since the phase angle is the arctangent of the ratio of the imaginary to the real component of equation (119).

The numerator of the real component of equation (119) reduces to

$$R_1^2\left(\frac{\omega^2/\omega_0^2 + 1}{\omega^2/\omega_0^2}\right) + 2R_1R_2\left(\frac{M - 1}{\omega^2/\omega_0^2 + 1}\right) + R_2^2\left(\frac{-M}{\omega^2/\omega_0^2 + 1}\right) \tag{120}$$

and the numerator of the imaginary component of equation (119), exclusive of j, reduces to

$$R_1R_2(1+M)\left(\frac{\omega/\omega_0}{\omega^2/\omega_0^2+1}-\frac{1}{\omega/\omega_0(\omega^2/\omega_0^2+1)}\right). \qquad (121)$$

With suitable simplifications, the expression for the phase angle of equation (119) becomes

$$\theta=\tan^{-1}\frac{R_2/R_1(1+M)(\omega^3\omega_0-\omega\omega_0^3)}{(\omega^4-2\omega^2\omega_0^2+\omega_0^4)-\omega^2\omega_0^2\left[M\,R_2^2/R_1^2-2(M-1)R_2/R_1-4\right]} \qquad (122)$$

or

$$\theta=\tan^{-1}\frac{R_2/R_1(1+M)(f^3f_0-ff_0^3)}{(f^4-2f^2f_0^2+f_0^4)-f^2f_0^2\left[M\,R_2^2/R_1^2-2(M-1)R_2/R_1-4\right]}. \qquad (123)$$

In order for equation (123) to be of the form

$$\theta=\tan^{-1}\frac{2s(f^3f_0-ff_0^3)}{(f^4-2f^2f_0^2+f_0^4)-S^2f^2f_0^2}, \qquad (124)$$

the following conditions must be imposed:

$$\left[R_2/R_1\left(\frac{1+M}{2}\right)\right]^2=\left[M\,R_2^2/R_1^2-2(M-1)R_2/R_1-4\right], \qquad (125)$$

in which the first bracketed term must equal s, and the last bracketed term must equal s^2. After transposing terms and clearing fractions, equation (125) reduces to the quadratic equation in M,

$$M^2 + M\left(-2 + 8\,R_1/R_2\right) + \left(1 - 8\,R_1/R_2 + 16\,R_1^2/R_2^2\right) = 0. \qquad (126)$$

The solution for M is

$$M = 1 - 4\,R_1/R_2. \qquad (127)$$

By letting the ratio $R_1/R_2 = C_2/C_1 = a''$,

$$M = 1 - 4a''. \qquad (128)$$

Since from equations (123) and (124),

$$2s = R_2/R_1\,(1 + M) \qquad (129)$$

$$s = \frac{1 - 2a''}{a''} \qquad (130)$$

$$a'' = \frac{1}{s + 2} \qquad (131)$$

$$M = \frac{s - 2}{s + 2}, \qquad (132)$$

the relationship

$$(1-4a'') = M = \frac{R_3}{R_2 + R_3} \qquad (133)$$

yields

$$R_3/R_2 = \frac{1 - 4a''}{4a''} \qquad (134)$$

and since $C_1R_1 = C_2R_2 = C_3R_3$ and $R_2 = R_1/a''$, the following equations are readily apparent:

$$R_1/R_3 = \frac{4a''^2}{1-4a''} \qquad (135)$$

$$C_3/C_1 = \frac{4a''^2}{1-4a''} \qquad (136)$$

$$C_2/C_3 = \frac{1-4a''}{4a''} \qquad (137)$$

$$C_2/C_1 = a''. \qquad (138)$$

The constant M is the fraction of one of the equal input voltages which appears across the output. The value of M shown in equation (132) is an indication of the effect of the choice of s on the magnitude of the output voltage. As s increases in value, M increases. In any event, s must be breater that 2; otherwise, the output will be zero for s equal to 2 and mathematically negative for s less than 2.

As in the L-C network, the f_o is the frequency at which θ equals 180°. For a value of s equal to 4 and using Dome's method for determining g, the phase-shift sections of two R-C channels (one critical) frequency each) may be determined by the following suggested method:

(1) Select a value for f_m and determine $F_m = \log_{10} f_m$.

(2) Add .328 to F_m to obtain F_{oB}; subtract .328 from F_m to obtain F_{oA}.

(3) Determine f_{oA} and f_{oB} from the relationship $10^F = f$.

For network A:

(4) Select a suitable value of R_{1A}. (Note: R_{1A} partially determines plate load of the vacuum tube.)

(5) R_{2A} then equals R_1/a'' where a'' equals $1/(s+2) = 1/6$.

(6) R_{3A} then equals $R_{2A}(1-4a''/4a'') = R_{2A}(\frac{1}{2})$.

(7) Determine C_{1A}, C_{2A} and C_{3A} from the relationships

$\omega_{oA} = 1/R_{1A}C_{1A} = 1/R_{2A}C_{2A} = 1/R_{3A}C_{3A}$.

For network B:

(8) Employ identical values for R_{1B}, R_{2B} and R_{3B}, respectively, as were determined for R_{1A}, R_{2A} and R_{3A}.

(9) Determine C_{1B}, C_{2B} and C_{3B} from the relationship

$\omega_{oB} = 1/R_{1B}C_{1B} = 1/R_{2B}C_{2B} = 1/R_{3B}C_{3B}$.

PART II

EXTENSION OF NETWORKS

ANALYSIS OF TWO POLE L-C NETWORK

It was observed in the previous sections that single pole L-C networks and single critical frequency R-C networks can be determined to give identical frequency-phase characteristics. When two such networks form channels A and B, the relative phase difference is approximately 90° over a specified range. By increasing the value of s, Dome[12] predicts that the variation from a 90° difference will be reduced. The range of operation, however, cannot be appreciably increased by an increase in s. This fact may be verified qualitatively, at least, by observing that at f sufficiently removed from f_o, a change in s causes very little change in θ, indicating that the effectiveness of s is confined to a region close to and symmetrical (on a logarithmic basis) about f_o.

As the frequency is varied from $-\infty$ to $+\infty$, the phase angle of a one pole L-C network varies from $\theta = 0^\circ$ to $\theta = 360^\circ$. It is conceivable that two networks, the phase angles of each varying from $\theta = 0^\circ$ to $\theta = 720^\circ$, may be used as channels A and B to yield an output which will retain the approximate 90° phase difference over an increased range. The extension of the original networks is based on this philosophy.

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[12]Dome, R. B., loc. cit.

The Z_1 impedance arm in the one pole L-C network is a special case of the network shown in Figure 9(b). Likewise, the Z_2 impedance arm is a special case of the network shown in Figure 9(c). Except for greater complexity of structure, the same general types of impedance arms are employed in the two pole L-C network as in the one pole network.

A schematic diagram of the two pole L-C network is shown in Figure 10. By comparing the Z_1 impedance arm with the illustration of Figure 9(b), it is seen that Z_1 has two poles of impedance. The term "two pole" in the network description stems from this fact.

The solution of the two pole L-C network closely parallels the solution of the one pole L-C network by the method of poles and zeros.

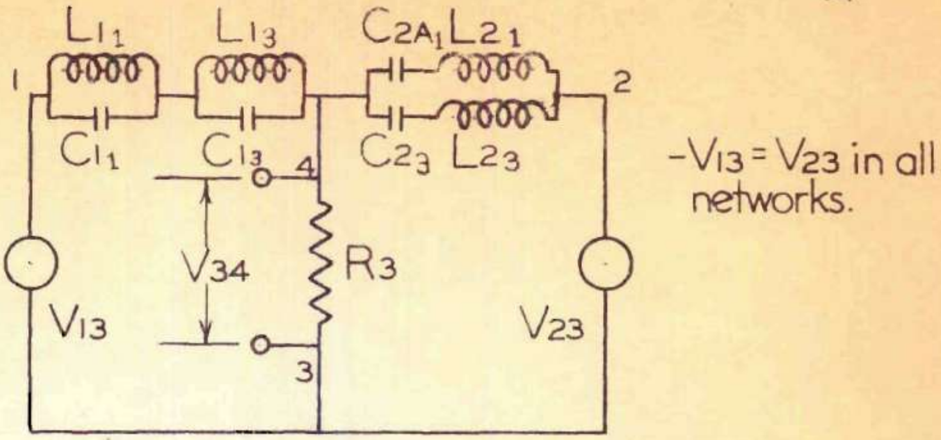
The general solution of Appendix I applies. The result of this solution is repeated below for clarity.

$$V_{34} = V_{13} \frac{(Z_1 - Z_2)}{\frac{Z_1 Z_2}{Z_3} + (Z_1 + Z_2)}. \qquad (139)$$

The impedance arm Z_1 has the Foster representation

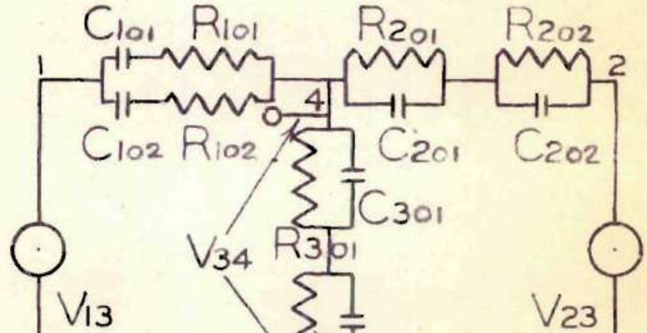
$$Z_1 = \frac{p}{C_1} \frac{(p^2 - p_2^2)}{(p^2 - p_1^2)(p^2 - p_3^2)}. \qquad (140)$$

At $p = p_2$, Z_1 has a zero of impedance (or a pole of admittance), and at



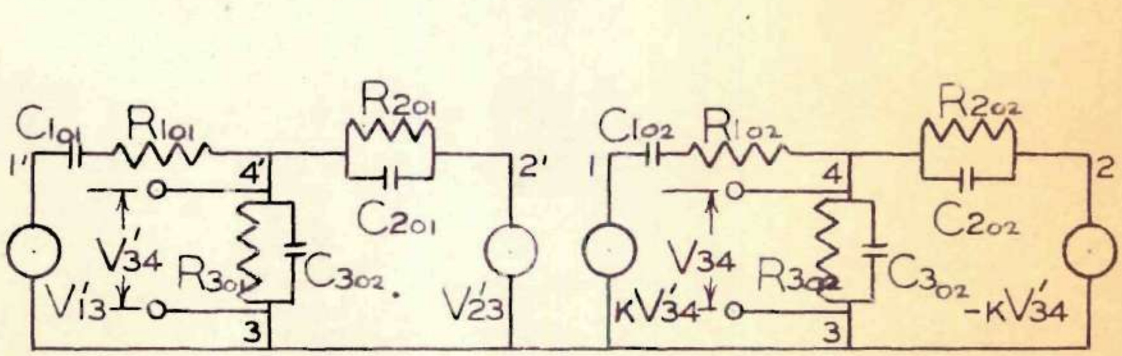
MESH FORM OF SINGLE TWO POLE L-C NETWORK.

FIG. 10



MESH FORM OF PROPOSED SINGLE R-C NETWORK WITH THREE CRITICAL FREQUENCIES.

FIG. 11



MESH FORM OF FINAL DOUBLE R-C NETWORK HAVING A TOTAL OF THREE CRITICAL FREQUENCIES COMPOSED OF TWO SINGLE R-C NETWORKS HAVING ONE CRITICAL FREQUENCY EACH.

FIG. 12

$p = p_1$ and $p = p_3$, Z_1 has poles of impedance. The sketch of Figure 9(b) is an identical representation of the frequency-reactance characteristics of this arm. C_1 is the equivalent capacitance at infinite frequencies. Since an inductance has infinite impedance at an infinite frequency, reference to Figure 10 shows that

$$C_1 = \frac{C_{1_1} C_{1_3}}{C_{1_1} + C_{1_3}}. \qquad (141)$$

The equivalent inductance at zero frequency is

$$L_1 = L_{1_1} + L_{1_3}. \qquad (142)$$

The impedance arm Z_2 may be represented as

$$Z_2 = \frac{L_2}{p} \frac{(p^2 - p_1^2)(p^2 - p_3^2)}{(p^2 - p_2^2)}. \qquad (143)$$

The zeros of impedance in Z_2 occur at the same value of p as do the poles of impedance in Z_1. Likewise, the zero of impedance in Z_1 occurs at the same value of p as does the pole of impedance in Z_2. This is similar to the single pole solution where the zero of impedance of Z_2 occurred at the same value of p, which caused the pole of impedance in Z_1. Should a three or higher pole L-C network be considered, the same equivalence would occur. Returning to Figure 10, it is seen that the equivalent inductance, L_2, at an infinite frequency is

$$L_2 = \frac{L_{2_1} L_{2_3}}{L_{2_1} + L_{2_3}} \tag{144}$$

and at zero frequency

$$C_2 = C_{2_1} + C_{2_3}. \tag{145}$$

As before

$$Z_3 = R_3. \tag{146}$$

Substitution of the values of equations (140), (143) and (146) in equation (139) yields

$$V_{34} = V_{13} \frac{p^2(p^2-p_2^2)^2 - L_2C_1(p^2-p_1^2)^2(p^2-p_3^2)^2}{L_2/R_3 \cdot (p^2-p_1^2)(p^2-p_2^2)(p^2-p_3^2)\,p + p^2(p^2-p_2^2) + L_2C_1(p^2-p_1^2)^2(p^2-p_3^2)^2}. \tag{147}$$

In the solution for the one pole L-C network by the method of zeros and poles, it is seen that

$$4R_3^2 = L_2/C_1 \tag{148}$$

where L_2 is the equivalent inductance of Z_2 and C_1 the equivalent capacitance of Z_1 at an infinite frequency. The same analogy is utilized in the two pole case. By letting

$$4R_3^2 = L_2/C_1;\ L_2 = 4R_3^2 C_1 \tag{149}$$

the expression for output voltage becomes

$$V_{34} = V_{13} \frac{p^2(p^2-p_2^2)^2 - 4R_3^2C_1^2(p^2-p_1^2)^2(p^2-p_3^2)^2}{4R_3C_1(p^2-p_1^2)(p^2-p_2^2)(p^2-p_3^2)\,p + p^2(p^2-p_2^2)^2 + 4R_3^2C_1^2(p^2-p_1^2)^2(p^2-p_3^2)^2}. \tag{150}$$

By noting that the numerator of equation (150) contains no imaginary terms and that the first expression of the denominator embraces all the imaginary terms, the negative of the phase angle of V_{34} is obtained by using the reciprocal of equation (150) and evaluating $-\theta$ as the arc-tangent of the ratio of the imaginary portion to the real portion. There results

$$-\theta = \tan^{-1} \frac{4\omega R_3C_1(p^2-p_1^2)(p^2-p_2^2)(p^2-p_3^2)}{4R_3^2C_1^2(p^2-p_1^2)^2(p^2-p_3^2)^2 + p^2(p^2-p_2^2)^2} \tag{151}$$

and

$$\theta = \tan^{-1} \frac{-4\omega R_3C_1(p^2-p_1^2)(p^2-p_2^2)(p^2-p_3^2)}{4R_3^2C_1^2(p^2-p_1^2)^2(p^2-p_3^2)^2 + p^2(p^2-p_2^2)^2}. \tag{152}$$

By substituting $j\omega = p$, $j\omega_1 = p_1$, etc., in equation (152) and rearranging terms, θ becomes

$$\theta = \tan^{-1}\left\{\frac{2\left(\frac{1}{2R_3C_1}\cdot\frac{1}{\omega_2}\right)\left[\omega^7\omega_2 - \omega^5\left(\omega_1^2\omega_2 + \omega_2^3 + \omega_2\omega_3^2\right) + \omega^3\left(\omega_1^2\omega_2\omega_3^2 + \omega_1^2\omega_2^3 + \omega_2^3\omega_3^2\right) - \omega\left(\omega_1^2\omega_2^3\omega_3^2\right)\right]}{\left[\omega^8 - \omega^6\left(2\omega_1^2 + 2\omega_3^2\right) + \omega^4\left(\omega_1^4 + 4\omega_1^2\omega_3^2 + \omega_3^4\right) - \omega^2\left(2\omega_1^4\omega_3^2 + 2\omega_1^2\omega_3^4\right) + \omega_1^4\omega_3^4\right] - \left(\frac{1}{4R_3^2C_1^2}\cdot\frac{1}{\omega_2^2}\right)\left(\omega^6\omega_2^2 - 2\omega^4\omega_2^4 + \omega^2\omega_2^6\right)}\right\}. \quad (153)$$

The numerator and denominator of equation (153) may be factored to give

$$\theta = \tan^{-1}\left\{\frac{2\left(\frac{1}{2R_3C_1\omega_2}\right)\left[\omega^3\omega_2 - \omega\omega_2^3\right]\left[\omega^4 - \omega^2\left(\omega_1^2 + \omega_3^2\right) + \omega_1^2\omega_3^2\right]}{\left[\omega^4 - \omega^2\left(\omega_1^2 + \omega_3^2\right) + \omega_1^2\omega_3^2\right]^2 - \left(\frac{1}{2R_3C_1\omega_2}\right)^2\omega^2\omega_2^2\left(\omega^2 - \omega_2^2\right)^2}\right\}. \quad (154)$$

By allowing s to equal

$$S = \frac{1}{2R_3C_1\omega}, \quad (155)$$

letting ω_2 be the geometric mean between ω_1 and ω_3 ($\omega_2^2 = \omega_1\omega_3$) and clearing fractions, the expression for θ becomes

$$\theta = \tan^{-1}\left\{\frac{2s\left(\omega^3\omega_2 - \omega\omega_2^3\right)}{\left[\omega^4 - \omega^2\left(\omega_1^2 + \omega_3^2\right) + \omega_2^4\right] - \dfrac{s^2\omega^2\omega_2^2\left(\omega^2 - \omega_2^2\right)^2}{\left[\omega^4 - \omega^2\left(\omega_1^2 + \omega_3^2\right) + \omega_2^4\right]}}\right\}. \quad (156)$$

By substituting the relationship $\omega = 2\pi f$,

$$\theta = \tan^{-1}\left\{\frac{2s\left(f^3 f_2 - f f_2^3\right)}{\left[f^4 - f^2\left(f_1^2 + f_3^2\right) + f_2^4\right] - \dfrac{s\, f^2 f_2^2 \left(f^2 - f_2^2\right)^2}{\left[f^4 - f^2\left(f_1^2 + f_3^2\right) + f_2^4\right]}}\right\} \qquad (157)$$

and since $f = 10^F$,

$$\theta = \tan^{-1}\left\{\frac{2s\left(10^{3F+F_2} - 10^{F+3F_2}\right)}{\left[10^{4F} - 10^{2F+2F_1} - 10^{2F+2F_2} + 10^{4F_2}\right] - \dfrac{s^2 \cdot 10^{2F+2F_2}\left(10^{2F} - 10^{2F_2}\right)^2}{\left[10^{4F} - 10^{2F+2F_1} - 10^{2F+2F_3} + 10^{4F_2}\right]}}\right\}. \qquad (158)$$

It is seen that f_2 occupies somewhat the same position in equation (157) as does f_o in equation (34). Should the points represented by f_1 and f_3 be equal to f_2, equation (157) will reduce to equation (34), with f_2 in place of f_o.

DETERMINATION OF S FOR TWO POLE L-C NETWORK

If equations (156) through (158) are to be useful, a suitable value of s must be selected such that the straight portion of the $\log_{10} f$ versus θ curve extends over a wide frequency range. Since ω_2 is the geometric mean between ω_1 and ω_3, f_2 is the geometric mean between f_1 and f_3. According to the previously developed relations, $2F_2 = F_1 + F_3$. At F equal to $-\infty$, F_1, F_2, F_3 or $+\infty$, θ equals $\tan^{-1} 0$. As F progresses from $-\infty$ to $+\infty$, θ varies from $0°$ to $720°$, and F_1, F_2 and F_3 are those values of F at which θ equals $180°$, $360°$ and $540°$, respectively. A straight line drawn between θ equal to $180°$ at F_1

and 540° at F_3 passes through θ equal to 360° at F_2. The slope of the straight line is $180/(F_2-F_1)$ degrees per unit of F. The straight line passes through 270° and -450° at $F=(F_1+F_2)/2$ and $F=(F_2+F_3)/2$, respectively. Should it be possible to determine the constants of equation (158) such that θ equals the above values in degrees at the corresponding F's for the straight line, at least five points would be secured to yield an average slope line. There would be no assurance, however, that any other points on the curve would coincide with points on the average slope line. The principal object is to determine a value of F_1 and F_3, together with a reasonable value of s, such that the actual deviation from the desired slope line is within allowable limits.

If the curve of F versus θ is to pass through the points $((F_1+F_2)/2, -270°)$ and $((F_2+F_3)/2, -450°)$, tan θ must be infinite at $F=(F_1+F_2)/2$ and $F=F_2+F_3)/2$. To satisfy these conditions, the denominator of

$$\tan\theta = \left\{ \frac{2s\left(10^{3F+F_2} - 10^{F+3F_2}\right)}{\left[10^{4F} - 10^{2F+2F_1} - 10^{2F+2F_3} + 10^{4F_2}\right] - \dfrac{S^2 \cdot 10^{2F+2F_2}\left(10^{2F}-10^{2F_2}\right)^2}{\left[10^{4F} - 10^{2F+2F_1} - 10^{2F+2F_3} + 10^{4F_2}\right]}} \right\} \qquad (159)$$

must equal zero at $F=(F_1+F_2)/2$ and $F=(F_2+F_3)/2$. Then

$$\left(10^{4F} - 10^{2F+2F_1} - 10^{2F+2F_3} + 10^{4F_2}\right) = S \cdot 10^{F+F_2}\left(10^{2F} - 10^{2F_2}\right) \qquad (160)$$

and

$$s = \frac{(10^{4F} - 10^{2F+2F_1} - 10^{2F+2F_2} + 10^{4F_2})}{10^{F+F_2}(10^{2F} - 10^{2F_2})} . \qquad (161)$$

For values of F_1 and F_3 such that (F_3-F_2), $(F_2-F_1) = 1$, s is in the order of 30. A spacing of .9 units of F is used in the final calculated and experimental networks in order to reduce the value of s.

On the basis that

$$F_3 - F_2 = .9 \;;\; F_3 = F_2 + .9 \qquad (162)$$

$$F_2 - F_1 = .9 \;;\; F_1 = F_2 - .9 , \qquad (163)$$

at

$$F = \frac{F_1 + F_2}{2} \qquad (164)$$

the expression for F becomes

$$F = F_2 - .45 . \qquad (165)$$

After substituting the F from equation (165), F_1 from equation (163) and F_3 from equation (162) in equation (161), s becomes

$$s = \frac{(10^{4F_2-1.8} - 10^{2F_2-.9+2F_2-1.8} - 10^{2F_2-.9+2F_2+1.8} + 10^{4F_2})}{10^{F_2-.45+F_2}(10^{2F_2-.9} - 10^{2F_2})} . \qquad (166)$$

By multiplying the numerator and denominator of equation (166) by $10^{(-4F_2+2.7)}$, there results

$$S = \frac{\left(10^{.9} - 1 - 10^{3.6} + 10^{2.7}\right)}{10^{1.35} - 10^{2.25}} = 22.345 \quad (166a)$$

from which

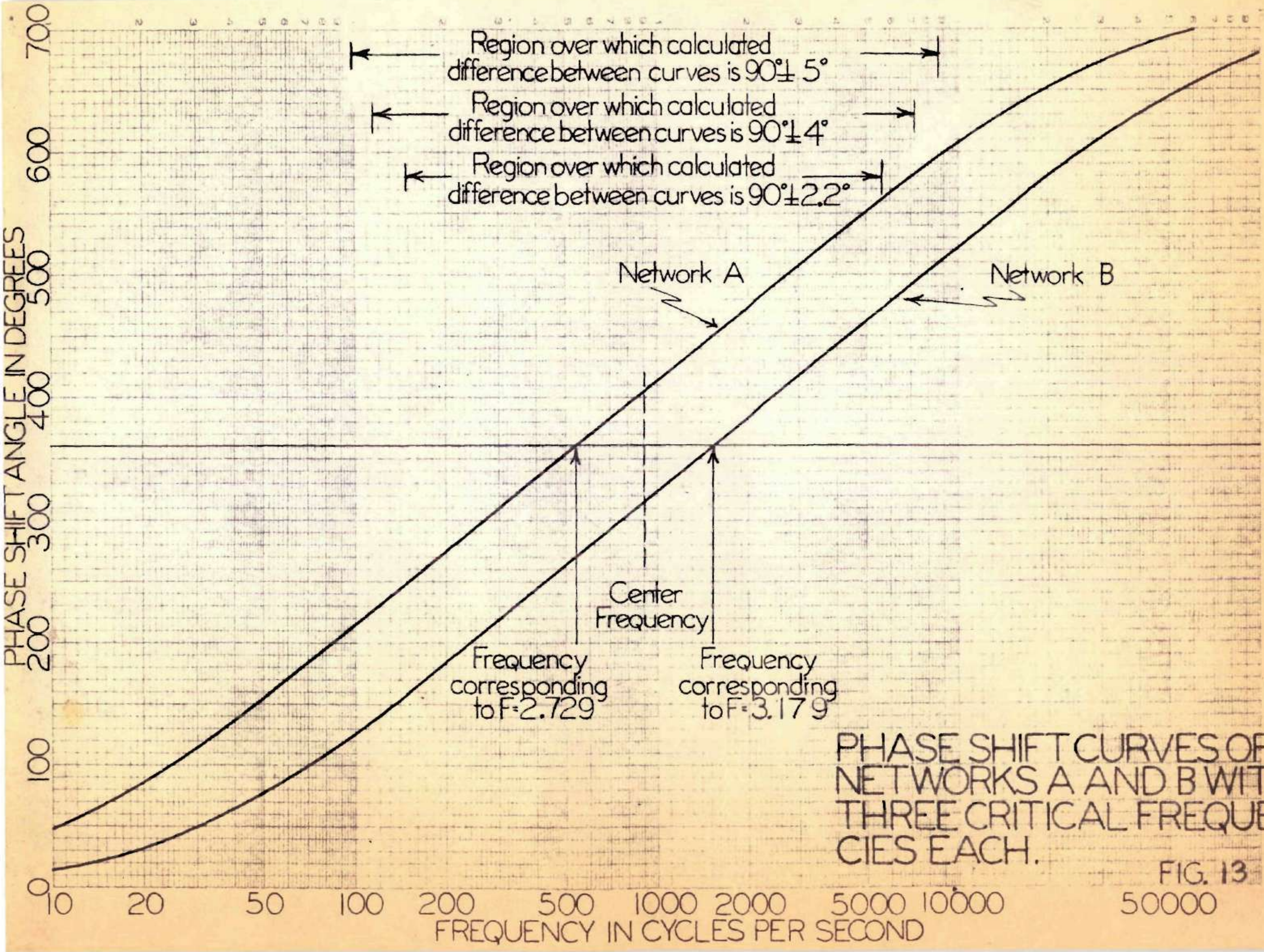
$$S^2 = 499.300. \qquad (167)$$

Appendix III lists the calculated values of θ determined by equation (158) with $s = 22.345$, $s^2 = 499.300$, $F_1 = 1.1$, $F_2 = 2.0$, $F_3 = 2.9$. The maximum difference between the calculated values of θ and the values of θ for the ideal slope line is 1.25° between $F = 1$ and $F = 3$. Since a curve of F (or $\log_{10}f$) versus θ may be laterally transposed and centered on other values of F_2, the data of Appendix III is used to plot the phase curves of Figure 13 and the error curves of Figure 14.

F_{2A} and F_{2B} (also f_{2A} and f_{2B}) are determined in the same general fashion for two pole L-C networks as F_{oA} and F_{oB} were determined for the one pole L-C networks or the one critical frequency R-C networks. The average slope line derived from the spacing of F_1, F_2 and F_3 determines the spacing of F_{2A} and F_{2B}. For

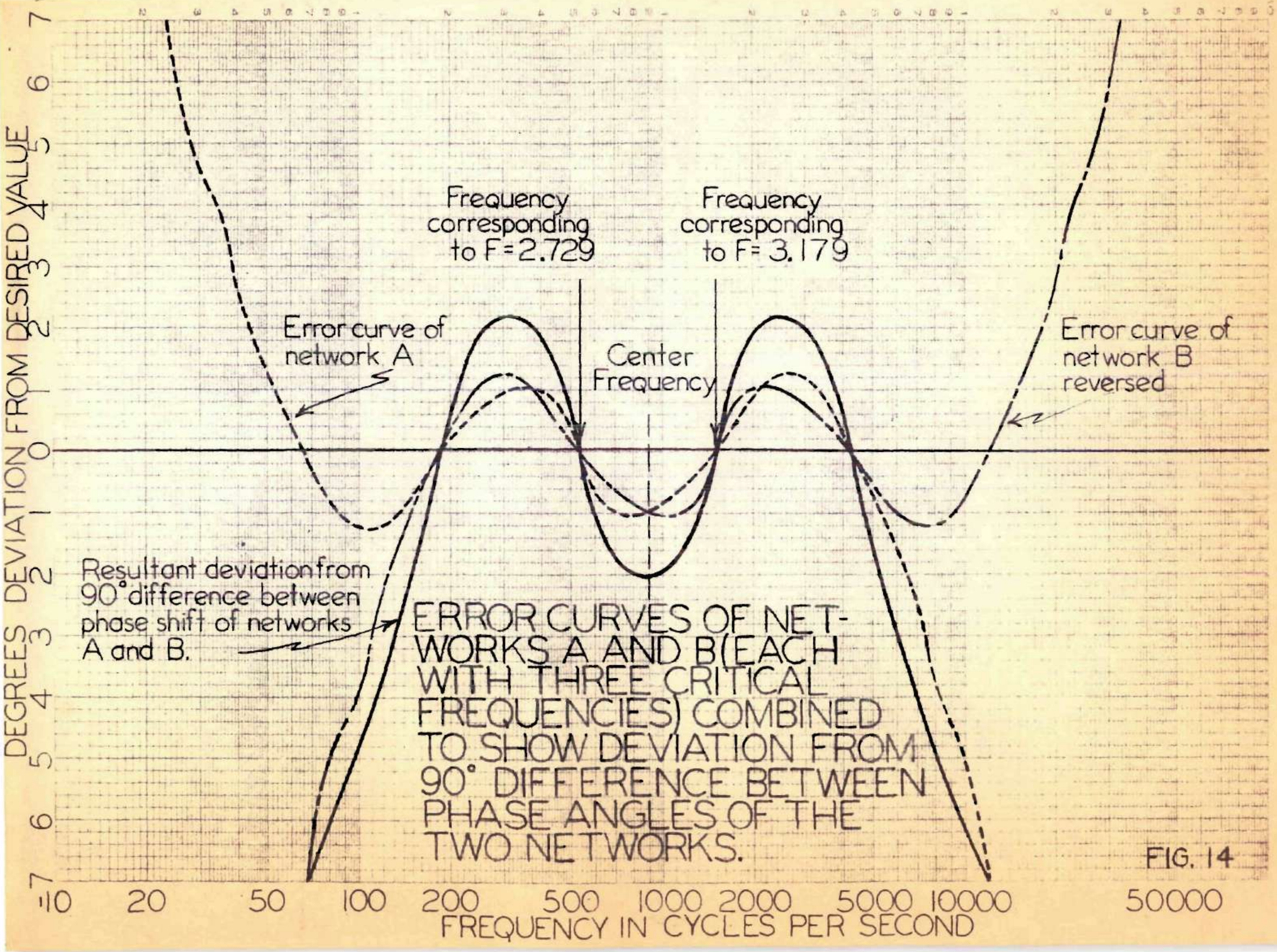
$$F_3 - F_2 = .9 \qquad (168)$$

$$F_2 - F_1 = .9\,, \qquad (169)$$



PHASE SHIFT CURVES OF NETWORKS A AND B WITH THREE CRITICAL FREQUENCIES EACH.

FIG. 13


DEGREES DEVIATION FROM DESIRED VALUE
FREQUENCY IN CYCLES PER SECOND
10 20 50 100 200 500 1000 2000 5000 10000 50000
7 6 5 4 3 2 1 0 1 2 3 4 5 6 7
Frequency corresponding to F=2.729
Frequency corresponding to F= 3.179
Error curve of network A
Center Frequency
Error curve of network B reversed
Resultant deviation from 90° difference between phase shift of networks A and B.
ERROR CURVES OF NETWORKS A AND B (EACH WITH THREE CRITICAL FREQUENCIES) COMBINED TO SHOW DEVIATION FROM 90° DIFFERENCE BETWEEN PHASE ANGLES OF THE TWO NETWORKS.
FIG. 14

the average slope becomes

$$\frac{180}{.9} = 200^\circ \text{ per unit of F.} \tag{170}$$

At F_M, the average slope line of network A passes through $\theta_A = 405^\circ$, while the average slope line of network B passes through $\theta_B = 315^\circ$. The difference between θ_A and θ_B is the required 90° difference. The equal ratios

$$\frac{45}{180} = \frac{g}{.9} \tag{171}$$

determine g as

$$g = .225 \tag{172}$$

where g equals the difference, in units of F, between F_M and either F_{2A} or F_{2B}. Once an F_M corresponding to the logarithm of the center frequency is selected,

$$F_{2A} = F_M - .225 \tag{173}$$

$$F_{2B} = F_M + .225. \tag{174}$$

The curves of Figures 13 and 14 illustrate the frequency-phase and error characteristics of the two channels, A and B, of a two pole L-C system. With s equal to 22.345, g equal to .225 and f_M equal to

900 cycles per second, the calculated maximum deviation from a 90^o phase difference is $\pm 2.2^o$ over a frequency range from 148 to 5600 cycles per second. Over the frequency range from 98 to 8600 cycles per second the calculated maximum deviation is $\pm 5^o$.

Should it be desired to obtain a pair of two pole L-C networks having phase characteristics similar to those shown in Figure 13, the following procedure is suggested:

(1) Select a value for f_M and determine $F_M = \log_{10} f_M$

(2) Add .225 to F_M to obtain F_{2B}; subtract .225 from F_M to obtain F_{2A}.

(3) Add .9 to F_{2A} to obtain F_{3A}; subtract .9 from F_{2A} to obtain F_{1A}.

(4) Add .9 to F_{2B} to obtain F_{3B}; subtract .9 from F_{2B} to obtain F_{1B}.

(5) Determine f_{1A}, f_{2A}, f_{3A}, f_{1B}, f_{2B}, f_{3B} from relationship $10^F = f$.

(6) Determine $\omega_{1A}, \omega_{2A}, \omega_{3A}, \omega_{1B}, \omega_{2B}, \omega_{3B}$ from relationship $\omega = 2\pi f$.

(7) Determine p^2_{1A}, p^2_{2A}, p^2_{3A}, p^2_{1B}, p^2_{2B}, p^2_{3B} from relationship $p = j\omega$.

For Channel A:

(8) Select a suitable value for R_{3A}.

(9) Determine C_{1A} (equivalent capacitance of impedance arm Z_1 in channel A at an infinite frequency) such that $C_1 = 1/2s\omega_{2A}R_{3A}$ or $C_1 = 1/44.69\omega_{2A}R_{3A}$.

(10) Determine L_{2A} (equivalent inductance of impedance arm Z_2 in channel A at an infinite frequency) such that $L_{2A} = 4R_{3A}^2C_{1A}$.

(11) Employ impedance reduction methods on $Z_{1A} = (p/C_{1A})(p^2-p_{2A}^2)/(p^2-p_{1A}^2)(p^2-p_{3A}^2)$ to obtain the circuit constants of the Z_1 impedance arm of channel A.

(12) Employ impedance reduction methods on $Z_{2A} = (L_{2A}/p)(p^2-p_{1A}^2)(p^2-p_{3A}^2)/(p^2-p_{2A}^2)$ to obtain the circuit constants of Z_2 impedance arm of channel A.

For Channel B:

(13) Repeat steps (8) through (12) as applicable to channel B.

R-C NETWORK WITH THREE CRITICAL FREQUENCIES - FIRST FORM

It was pointed out earlier that an R-C network is more practical than the L-C type. A direct correspondence between the one pole L-C network and the R-C network with one critical frequency was realized in a previous analysis. Identical equations for θ were obtained in either case. The one possible disadvantage of the R-C network is the attenuation encountered. However, attenuation may be overcome by amplification.

The R-C network shown in Figure 11 is a resistance analog of the two pole L-C network. An attempt will be made to determine under what conditions the frequency-phase characteristics of the network of Figure 11 are identical to those of the two pole L-C network.

It can be shown that the impedance arms of the R-C network of Figure 11 may be represented as

$$Z_1 = \frac{R_1}{p} \cdot \frac{(p+\omega_{01})(p+\omega_{02})}{(p+\omega_2)} \qquad (175)$$

$$Z_2 = \frac{1}{C_2} \cdot \frac{(p+\omega_2)}{(p+\omega_{01})(p+\omega_{02})} \qquad (176)$$

$$Z_3 = \frac{1}{C_3} \cdot \frac{(p+\omega_2)}{(p+\omega_{01})(p+\omega_{02})} \qquad (177)$$

where R_1, C_2 and C_3 are the equivalent resistance and capacitances of Z_1, Z_2 and Z_3, respectively, at infinite frequencies.[13] In addition,

$$\omega_{01} = \frac{1}{R_{1_{01}} C_{1_{01}}} = \frac{1}{R_{2_{01}} C_{2_{01}}} = \frac{1}{R_{3_{01}} C_{3_{01}}} \qquad (178)$$

$$\omega_{02} = \frac{1}{R_{1_{02}} C_{1_{02}}} = \frac{1}{R_{2_{02}} C_{2_{02}}} = \frac{1}{R_{3_{02}} C_{3_{02}}} \qquad (179)$$

and ω_2 equals some specified ω between ω_{01} and ω_{02} such that

$$\omega_{01} \leqslant \omega_2 \leqslant \omega_{02}. \qquad (180)$$

It should be noted that ω_{01} and ω_{02} are not specified as being equal to the ω_1 and ω_3 of the two pole L-C network. ω_1 and ω_3, in the two pole L-C network, are proportional to the frequencies at which θ equals 180° and 540°, respectively. ω_1 and ω_3 also are values of ω at which

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[13] Bode, H. W., op. cit., pp. 214-216.

poles and zeros of impedance occur. The Z_1 impedance arm of the R-C network, shown in Figure 11, has a pole of impedance only at zero frequency. No true zero of impedance exists for this arm. On the other hand, the Z_2 and Z_3 impedance arms have zeros of impedance only at infinite frequency, but have no true poles of impedance. However, in accordance with the definition of a critical frequency, there are three critical frequencies associated with each of the impedances Z_1, Z_2 or Z_3 of Figure 11. As a specific example, the Z_1 impedance arm has critical frequencies when the voltage drops across each element of the R_{101} C_{101} series combination are equal, when the voltage drops across each element of the R_{102} C_{102} series combination are equal and when the absolute value of the reactive component of the entire Z_1 impedance equals the resistive component. The critical frequencies are proportional to ω_{01}, ω_{02} and ω_2 in the order given.

By substituting the values of Z_1, Z_2 and Z_3 from equations (175) through (177) in equation (139), the following relationship is obtained:

$$V_{34} = MV_{13}\left\{\frac{\dfrac{R_1}{p}\cdot\dfrac{(p+\omega_{01})(p+\omega_{02})}{(p+\omega_2)} - \dfrac{1}{C_2}\dfrac{(p+\omega_2)}{(p+\omega_{01})(p+\omega_{02})}}{\dfrac{R_1}{p}\cdot\dfrac{(p+\omega_{01})(p+\omega_{02})}{(p+\omega_2)} + \dfrac{M}{C_2}\cdot\dfrac{(p+\omega_2)}{(p+\omega_{01})(p+\omega_{02})}}\right\}. \qquad (181)$$

The M in the above equation is

$$M = \frac{C_2}{C_2 + C_3} \qquad (182)$$

and occupies the same position in equation (181) as

$$M = \frac{C_2}{C_2 + C_3} \tag{183}$$

in equation (117).

After substituting $j\omega = p$, clearing fractions and rearranging terms, equation (181) reduces to

$$V_{34} = MV_{13}\left\{\frac{\left[\left(\omega^4 - \omega^2\omega_{01}^2 - \omega^2\omega_{02}^2 + \omega_{01}^2\omega_{02}^2\right) + \left(2\frac{\omega^2\omega_2}{R_1C_1} - 4\omega^2\omega_{01}\omega_{02}\right)\right] + j\left[\left(-2\omega^3\omega_{01} - 2\omega^3\omega_{02} + 2\omega\omega_{01}^2\omega_{02} + 2\omega\omega_{01}\omega_{02}^2\right) + \left(\frac{\omega^3}{R_1C_2} - \frac{\omega\omega_2^2}{R_1C_2}\right)\right]}{\left[\left(\omega^4 - \omega^2\omega_{01}^2 - \omega^2\omega_{02} + \omega_{01}^2\omega_{02}^2\right) + \left(\frac{-2M\omega^2\omega_2}{R_1C_2} - 4\omega^2\omega_{01}\omega_{02}\right)\right] + j\left[\left(-2\omega^3\omega_{01} - 2\omega^3\omega_{02} + 2\omega\omega_{01}^2\omega_{02} + 2\omega\omega_{01}\omega_{02}^2\right) + (-M)\left(\frac{\omega^3}{R_1C_2} - \frac{\omega\omega_2^2}{R_1C_2}\right)\right]}\right\}. \tag{184}$$

The phase angle of V_{34} is the angle of the coefficient of MV_{13} in equation (184). By following the previously adopted convention of making ω_2 the geometric mean between ω_{01} and ω_{02}, it is possible to state

$$\omega_2^2 = \omega_{01}\omega_{02} \tag{185}$$

and make appropriate substitutions as desired. The phase angle, θ, of V_{34} is obtained by rationalizing the coefficient of MV_{13} in equation (184) and finding the arctangent of the ratio of the coefficient of the imaginary to the real component. There results

$$\theta=\tan^{-1}\left\{\frac{\begin{array}{l}2\left(\frac{1+M}{2\omega_2R_1C_2}\right)\{\omega^7\omega_2-\omega^5\omega_2[\omega_2^2+\omega_{01}^2+\omega_{02}^2+4\omega_2^2-4\omega_2(\omega_{01}+\omega_{02})] \\ +\omega^3\omega_2[\omega_2^4+\omega_2^2\omega_{01}^2+\omega_2^2\omega_{02}^2+4\omega_2^4-4\omega_2^3(\omega_{01}+\omega_{02})]-\omega\omega_2^7\}\end{array}}{\begin{array}{l}\{\omega^8-\omega^6[8\omega_2^2+2\omega_{01}^2+2\omega_{02}^2-2\frac{\omega_2}{R_1C_2}(1-M)]+\omega^4[20\omega_2^4 \\ +8\omega_2^2(\omega_{01}^2+\omega_{02}^2)-2\frac{\omega_2}{R_1C_2}(1-M)(\omega_{01}^2+\omega_{02}^2)+ \\ (\omega_{01}^4+\omega_{02}^4)-4\frac{M\omega_2^2}{R_1^2C_2^2}-\frac{8\omega_2^3}{R_1C_2}(1-M)]-\omega^2[8\omega_2^6+2\omega_2^4\omega_{01}^2 \\ 2\omega_2^4\omega_{02}^2-2\omega_2^5(1-M)]+\omega_2^8\}-[\frac{M}{\omega_2^2R_1^2C_2^2}+ \\ 2\frac{(1-M)}{\omega_2R_1C_2}\left(\frac{\omega_{01}+\omega_{02}}{\omega_2}\right)-4\left(\frac{\omega_{01}+\omega_{02}}{\omega_2}\right)^2](\omega^6\omega_2^2 \\ -2\omega^4\omega_2^4+\omega^2\omega_2^6)\end{array}}\right\}. \tag{186}$$

Equation (153) for the phase angle of the two pole L-C network becomes

$$\theta=\tan^{-1}\left\{\frac{\begin{array}{l}2s[\omega^7\omega_2-\omega^5\omega_2(\omega_1^2+\omega_2^2+\omega_3^2)+ \\ \omega^3\omega_2(\omega_1^2\omega_2^2+\omega_1^2\omega_3^2+\omega_2^2\omega_3^2)-\omega\omega_2(\omega_2^6)\end{array}}{\begin{array}{l}[\omega^8-\omega^6(2\omega_1^2+2\omega_3^2)+\omega^4(\omega_1^4+4\omega_2^4+\omega_3^4) \\ -\omega^2(2\omega_1^4\omega_3^2+2\omega_1^2\omega_3^4)+\omega_1^4\omega_3^4] \\ -s^2(\omega^6\omega_2^2-2\omega^4\omega_2^4+\omega^2\omega_2^6)\end{array}}\right\} \tag{187}$$

when $1/2\omega R_3C_1$ and $\omega_1\omega_3$ are replaced by their equivalents, s and ω_2^2, respectively. By equating like terms of equations (186) and (187), the conditions under which equation (186) represents the desired θ are determined. It is immediately seen that

$$S = \frac{(1 + M)}{2\omega_2 R_1 C_2}. \qquad (188)$$

Squaring equation (188) yields

$$S^2 = \frac{(1 + M)^2}{4\omega_2^2 R_1^2 C_2^2}. \qquad (189)$$

From the donominators of equations (186) and (187), it is apparent that

$$S^2 = \left[\frac{M}{\omega_2^2 R_1^2 C_2^2} + \frac{2(1-M)}{\omega_2 R_1 C_2}\left(\frac{\omega_{01} + \omega_{02}}{\omega_2}\right) - 4\left(\frac{\omega_{01} + \omega_{02}}{\omega_2}\right)^2\right]. \qquad (199)$$

Equating the right hand members of equations (189) and (190) provides

$$M = 1 - R_1 C_2 (\omega_{01} + \omega_{02}). \qquad (191)$$

By substituting the above value of M in the coefficient of the first ω^6 term of the denominator in equation (186) and equating the result to the corresponding term of equation (187), it is seen that

$$\omega_{01}^2 + \omega_{02}^2 - 4\omega_2(\omega_{01} + \omega_{02}) + 4\omega^2 = (\omega_1^2 + \omega_3^2). \qquad (192)$$

This expression also equates the coefficients of the corresponding ω^5 and ω^3 terms of the numerators and the coefficients of the ω^2 terms of the denominators. The coefficients of the ω^8, ω^7, and ω_2 terms in each equation are directly equal. The derived term for s^2 in equation (190) has the same coefficient in equation (186) as s^2 in equation (187). A direct equivalence has thus been established between all terms of equations (186) and (187), with the exception of the first ω^4 term in the denominators.

The coefficient of the ω^4 term of equation (187) may be expressed as

$$\left(\omega_1^4 + 4\omega_2^4 + \omega_3^4\right) = \left(\omega_1^2 + \omega_3^2\right)^2 + 2\omega_2^4. \qquad (193)$$

By substituting for $(\omega_1^2 \omega_3^2)$ its value from equation (192) and letting $\omega_2^2 = \omega_{01}\omega_{02}$, there results

$$\begin{aligned}(\omega_1^2 + \omega_3^2)^2 + 2\omega_2^4 &= 20\omega_2^4 - 32\omega_2^3(\omega_{01} + \omega_{02}) \\ &+ 16\omega_2^2(\omega_{01} + \omega_{02})^2 + 8\omega_2^2(\omega_{01}^2 + \omega_{02}^2) - 8(\omega_{01} + \omega_{02})(\omega_{01}^2 + \omega_{02}^2) \\ &+ (\omega_{01}^4 + \omega_{02}^4). \end{aligned} \qquad (194)$$

By substituting the value of M determined by equation (191), in the coefficient of the ω^4 term of equation (186), the ω^4 coefficient becomes

$$\begin{aligned}&20\omega_2^4 - 32\,\omega_2^3(\omega_{01} + \omega_{02}) - \frac{4\omega_2^2}{R_1^2 C_2^2} + 16\frac{\omega_2^2}{R_1 C_2}(\omega_{01} + \omega_{02}) \\ &+ 8\omega_2^2(\omega_{01}^2 + \omega_{02}^2) - 8\omega_2(\omega_{01} + \omega_{02})(\omega_{01}^2 + \omega_{02}^2) \\ &+ (\omega_{01}^4 + \omega_{02}^4). \end{aligned} \qquad (195)$$

It is apparent that the right side of formula (194) and expression (195) are equal if

$$16\omega_2^2\left(\omega_{01}+\omega_{02}\right)^2 = \frac{16\omega_2^2}{R_1C_2}\left(\omega_{01}+\omega_{02}\right) - \frac{4\omega_2^2}{R_1^2C_2^2} \qquad (196)$$

or

$$\left(\omega_{01}+\omega_{02}\right) = \frac{1}{2R_1C_2}. \qquad (197)$$

The conditions for complete equivalence between equations (186) and (187) may now be grouped as follows:

$$M = \frac{C_2}{C_2 + C_3} \qquad (198)$$

$$S = \frac{(1+M)}{2\omega_2 R_1 C_2} \qquad (199)$$

$$M = 1 - 4R_1C_2\left(\omega_{01}+\omega_{02}\right) \qquad (200)$$

$$\left(\omega_{01}^2+\omega_{02}^2\right) - 4\omega_2\left(\omega_{01}+\omega_{02}\right) + 4\omega_2^2 = \left(\omega_1^2+\omega_3^2\right) \qquad (201)$$

$$\left(\omega_{01}+\omega_{02}\right) = \frac{1}{2R_1C_2}. \qquad (202)$$

On the surface, it appears that equivalence may be obtained. M is identical to the expression for the M of the R-C network with one critical frequency. The parameter s for the single critical frequency case is

$$S = \frac{R_2}{2R_1}\left(1+M\right) = \frac{1}{2\omega_0 R_1 C_2}\left(1+M\right) \qquad (203)$$

which bears a resemblance to the s of equation (199). The M of the one critical frequency R-C network is equal to

$$M = 1 - 4\,R_1/R_2 = 1 - 4R_1C_2\omega_0 \qquad (204)$$

which has a similar form to the M in equation (200). However, when the value of $(\omega_{01}\omega_{02})$ in equation (197) is substituted in equation (200), M becomes

$$M = 1 - \frac{4R_1C_2}{2R_1C_2} = -1. \qquad (205)$$

This condition requires that either C_2 be negative and less than C_3, or that C_3 be negative and greater in absolute value than C_2. Either condition is incompatible with the solution desired, since only positive circuit elements are to be used. Thus, a solution of the R-C network of Figure 11 is mathematically possible, but physically impractical.

From a comparison of the two pole L-C network, which provided the desired phase characteristics, and the R-C network just considered, it is deduced that a successful R-C network must possess the same degree of isolation between phase shift parameters as is provided by the distinct poles and zeros of the impedance arms in the two pole L-C network. This isolation is not provided by the circuit of Figure 11.

ISOLATED TYPE OF R-C NETWORK WITH THREE CRITICAL FREQUENCIES

The obvious conclusion which is drawn from the mathematical results of the previous section is that an extended R-C type of network requires some form of isolation within the network. Dome's[14] second network suggests a means for isolation by employing a vacuum tube between sets of circuit elements. It is shown in this section that two R-C networks, each with one critical frequency, may be combined by using a vacuum tube as both the isolating device and the connecting link between the two networks. The characteristics of the two R-C networks are chosen with a different end in view than was sought when this type of network was first analyzed. No attempt is made to fit the frequency-phase curves to an average slope line. Rather, the method of attack in this section is to determine the individual characteristics of two networks, such that the combined effect duplicates the frequency-phase characteristics of a two pole L-C network. The term "three critical frequencies" is retained in describing the complete combination of the two R-C networks and the vacuum tube.

The phase angle expression, equation (187), of the two pole L-C network may be rearranged as

[14]Dome, R. B., loc. cit.

$$\theta = \tan^{-1}\left(\frac{2\left\{\omega^7(S\omega_2) - \omega^5\left[(S\omega_2)\omega_1^2 + (S\omega_2)\omega_2^2 + (S\omega_2)\omega_3^2\right] + \omega^3\left[(S\omega_2)\omega_1^2\omega_2^2 + (S\omega_2)\omega_1^2\omega_3^2 + (S\omega_2)\omega_2^2\omega_3^2\right] - \omega\left[(S\omega_2)\omega_2^6\right]\right\}}{\left\{\omega^8 - \omega^6\left[2\omega_1^2 + 2\omega_3^2\right] + \omega^4\left[\omega_1^4 + 4\omega_1^2\omega_3^2 + \omega_3^4\right] - \omega^2\left[2\omega_1^4\omega_3^2 + 2\omega_1^2\omega_3^4\right] + \omega_2^8\right\} - \left\{(S\omega_2)^2\cdot\omega^6 - (S\omega_2)^2\cdot 2\omega^4\omega_2^2 + (S\omega_2)^2\cdot\omega^2\omega_2^4\right\}}\right) \quad (206)$$

when the substitution $\omega_2^2 = \omega_1\omega_2$ is made.

The total phase angle derived from two single critical frequency R-C networks coupled by a vacuum tube, as shown in Figure 12, is

$$\Theta = \Theta_1 + \Theta_2 = \tan^{-1}\left\{\frac{2s_1\left(\omega^3\omega_{01} - \omega\,\omega_{01}^3\right)}{\left(\omega^4 - 2\omega^2\omega_{01}^2 + \omega_{01}^4\right) - s_1^2\left(\omega^2\omega_{01}^2\right)}\right\} + \tan^{-1}\left\{\frac{2s_2\left(\omega^3\omega_{02} - \omega\,\omega_{02}^3\right)}{\left(\omega^4 - 2\omega^2\omega_{02}^2 + \omega_{02}^4\right) - s_2^2\left(\omega^2\omega_{02}^2\right)}\right\} \quad (207)$$

where θ_1, s_1 and ω_{01} are the phase angle, the parameter s and the ω at which θ_1 equals 180°, respectively, of network 1, and where the subscript 2 refers to the similar values of network 2.

One form of the sum of two arctangents is[15]

$$\tan^{-1}u + \tan^{-1}v = \tan^{-1}\left(\frac{u + v}{1 - uv}\right). \quad (208)$$

[15] Hudson, R. G., op. cit., p. 12.

By applying equation (208) to equation (207), the expression for the total phase angle, when expanded and regrouped, becomes

$$\theta = \tan^{-1}\left\{\frac{\begin{array}{l} 2[\omega^7(s_1\omega_{01}+s_2\omega_{02}) - \omega^5(2s_1\omega_{01}\omega_{02}^2 + 2s_2\omega_{01}^2\omega_{02} + s_1\omega_{01}^3 \\ + s_2\omega_{02}^3 + s_1^2 s_2\omega_{01}^2\omega_{02} + s_1 s_2^2\omega_{01}\omega_{02}^2) + \omega^3(s_1\omega_{01}\omega_{02}^4 \\ + s_2\omega_{01}^4\omega_{02} + 2s_1\omega_{01}^3\omega_{02}^2 + 2s_2\omega_{01}^2\omega_{02}^3 + s_1^2 s_2\omega_{01}^2\omega_{02}^3 + \\ s_1 s_2^2\omega_{01}^3\omega_{02}^2) - \omega(s_1\omega_{01}^3\omega_{02}^4 + s_2\omega_{01}^4\omega_{02}^3)] \end{array}}{\begin{array}{l} \{\omega^8 - \omega^6[2\omega_{01}^2 + 2\omega_{02}^2 + 2s_1 s_2\omega_{01}\omega_{02}] + \omega^4[\omega_{01}^4 + \\ 4\omega_{01}^2\omega_{02}^2 + \omega_{02}^4 + 2s_1 s_2(\omega_{01}^3\omega_{02} + \omega_{01}\omega_{02}^3) + \\ s_1^2 s_2^2\omega_{01}^2\omega_{02}^2] - \omega^2[2\omega_{01}^4\omega_{02}^2 + 2\omega_{01}^2\omega_{02}^4 + 2s_1 s_2\omega_{01}^3\omega_{02}^3 \\ + \omega_{01}^4\omega_{02}^4]\} - \{\omega^6[s_1^2\omega_{01}^2 + s_2^2\omega_{02}^2] - \omega^4[4s_1^2\omega_{01}^2\omega_{02}^2 \\ + \omega^2[s_1^2\omega_{01}^2\omega_{02}^4 + s_2^2\omega_{01}^4\omega_{02}^2]\} + \{2s_1 s_2[\omega^6\omega_{01}\omega_{02} \\ - \omega^4(\omega_{01}^3\omega_{02} + \omega_{01}\omega_{02}^3) + \omega^2\omega_{01}^3\omega_{02}^3\} \end{array}}\right\}. \tag{209}$$

If equation (209) is to be of the required form, represented by equation (206), the coefficients of ω and the constant terms of equation (209) must equal the corresponding terms of equation (206). In equation (206), ω_2 is the geometric mean between ω_1 and ω_3.

Comparison of the constant term, $\omega_{01}^4\omega_{02}^4$, of the denominator of equation (209) with the corresponding term, ω_2^8, immediately fixes the condition

$$\omega_{01}\omega_{02} = \omega_2^2 . \tag{210}$$

Equality between the ω^7 coefficients of the numerators is expressed as

$$S\omega_2 = S_1\omega_{01} + S_2\omega_{02} \qquad (211)$$

Substitution of condition (210) in the ω coefficient of equation (209) and equating the result to the ω coefficient of equation (206) yields

$$S\omega_2 = S_2\omega_{01} + S_1\omega_{02}. \qquad (212)$$

In order for equation (212) to be reconciled with equation (211), the relationship

$$S = S_2 \qquad (213)$$

and

$$\frac{S\omega_2}{S_1} = \left(\omega_{01} + \omega_{02}\right) \qquad (214)$$

must apply.

One possible location of ω_{01} and ω_{02} with respect to ω_1, ω_2 and ω_3 is at ω_2, making $\omega_{01}\omega_{02} = \omega_2^2$. Although the resulting phase-shift angle progresses from 0° to 720°, the frequency-phase angle characteristics are not an improvement over the characteristics of a single critical frequency network. At any particular frequency, the phase angle is merely doubled, thereby indicating that ω_2 is not the desired location for ω_{01} and ω_{02}. At ω_1, the phase angle of the output voltage of the two pole L-C network is 180°. Since either θ_1 or θ_2 is greater than zero in the region under consideration, and since the sum of θ_1 and θ_2 is to be equal to 180° at ω_1, both θ_1 and θ_2 are less than 180° at ω_1. Therefore, the ω's at which θ_1 and θ_2 separately equal 180° are greater than ω_1. In a similar manner, it may be shown that the ω's at which θ_1 and θ_2 separately equal 180° are less than ω_3. This, together with the relationship $\omega_{01}\omega_{02} = \omega_2^2$, places ω_{01} between ω_1 and ω_2 and ω_{02} between ω_2 and ω_3. On a logarithmic basis, ω_{01} and ω_{02} are symmetrically placed with respect to ω_2.

By noting that $s_1 s_2 = s_1^2 = s_2^2$ and $\omega_{01}\omega_{02} = \omega_2^2$, the last two terms in braces in the denominator of equation (209) may be regrouped as

$$\omega^6\left[s_1(\omega_{01}+\omega_{02})\right]^2 - 2\omega^4\omega_2^2\left[s_1(\omega_{01}+\omega_{02})\right]^2 - \omega^2\omega_2^4\left[s_1(\omega_{01}+\omega_{02})\right]^2. \tag{215}$$

By virtue of $s_1 = s_2$, equation (212) reduces to

$$s_1(\omega_{01}+\omega_{02}) = \omega_2 s\,; \quad (\omega_{01}+\omega_{02}) = \frac{s\omega_2}{s_1}. \tag{216}$$

After substitution of the left member of equation (216), equation (215) reduces to

$$s^2\left(\omega^6\omega_2^2 - 2\omega^4\omega_2^4 + \omega^2\omega_2^6\right) \tag{217}$$

which is identical to the last term in the denominator of equation (206).

The coefficient of the remaining ω^6 term in equation (209) when equated to the corresponding term in equation (206) yields

$$\omega_{01}^2 + \omega_{02}^2 + S_1^2\omega_2^2 = \omega_1^2 + \omega_3^2 \qquad (218)$$

which is consistent with the relative magnitudes of ω_{01} and ω_{02} with respect to ω_1 and ω_3. By employing the relationship of equation (218), the coefficient of the remaining ω^2 term in the denominator of equation (209) reduces to the corresponding coefficient of equation (206). By rearranging terms and substituting $s_2 = s_1$ and $\omega_{01}\omega_{02} = \omega_2^2$, the coefficient of the ω^5 term in the numerator of equation (209) is factored to yield

$$S_1(\omega_{01}+\omega_{02})(\omega_{02}^2+\omega_{01}^2+S_1^2\omega_2^2)+S_1(\omega_{01}+\omega_{02})\omega_2^2. \qquad (219)$$

By substituting in expression (219) the relations of equations (216) and (218), expression (219) becomes

$$(S\omega_2)\omega_1^2 + (S\omega_2)\omega_2^2 + (S\omega_2)\omega_3^2 \qquad (220)$$

which is identical to the corresponding term of equation (206). The coefficients of the ω^3 terms in the numerators of equations (206) and (209) are equal to the coefficients of the ω^5 terms multiplied by ω_2^2. Hence, expression (220) applies equally well for these terms, and the desired correspondence is established.

By substituting $s_2 = s_1$ and $\omega_{01}\omega_{02} = \omega_2^2$ in the ω^4 coefficient in the denominator of equation (209) and rearranging terms, the ω^4 coefficient becomes

$$\left(\omega_{01}^2 + \omega_{02}^2 + s_1\omega_2^2\right)^2 + 2\omega_2^4 \qquad (221)$$

and, by equation (218), equals

$$\left(\omega_1^2 + \omega_3^2\right)^2 + 2\omega_2^4 \qquad (222)$$

which is identical to the coefficient of the ω^4 term in the denominator of equation (206). All coefficients of ω and all constant terms in equation (209) are now equated to the corresponding terms in equation (206), under the following conditions:

$$\omega_{01}\omega_{02} = \omega_2^2 \qquad (223)$$

$$s_1 = s_2 \qquad (224)$$

$$s_1\left(\omega_{01} + \omega_{02}\right) = s\omega_2 \, ; \; s_1 = s\left(\frac{\omega_2}{\omega_{01} + \omega_{02}}\right) \qquad (225)$$

$$\left(\omega_{01}^2 + \omega_{02}^2\right) = \omega_1^2 + \omega_3^2 - s_1^2\omega_2^2 . \qquad (226)$$

It is thus seen that with suitable restrictions placed on the circuit elements, two R-C networks with one critical frequency each may be combined to yield identical frequency-phase characteristics as were obtained with the two pole L-C network.

The parameter s_1 is an important factor in predicting the behavior of each one critical frequency R-C network. It is noted that the parameter s_1 of equation (225) is the s of each one critical frequency R-C network. Determination of s_1 in terms of the ω's follows:

At ω_1,

$$\theta = \theta_1 + \theta_2 = -180^\circ \tag{227}$$

$$\theta_1 = -\theta_2 - 180^\circ \tag{228}$$

$$-\theta_1 = \theta_2 + 180^\circ \tag{229}$$

and

$$\tan(-\theta_1) = -\tan\theta_1 = \tan\theta_2 . \tag{230}$$

The expressions for $-\tan\theta_1$ and $\tan\theta_2$ at ω_1 as determined from equation (207) are equated in the following manner:

$$-\frac{2s_1(\omega^3\omega_{01} - \omega\omega_{01}^3)}{(\omega^4 - 2\omega^2\omega_{01}^2 + \omega_{01}^4) - s_1^2(\omega^2\omega_{01}^2)} = \frac{2s_2(\omega^3\omega_{02} - \omega\omega_{02}^3)}{(\omega^4 - 2\omega^2\omega_{02}^2 + \omega_{02}^4) - s_2^2(\omega^2\omega_{02}^2)} . \tag{231}$$

By using the relation $s_1 = s_2$, combining and rearranging terms, equation (231) becomes

$$\begin{aligned}&\omega_1^7(\omega_{02} + \omega_{01}) - \omega^5(\omega_{01}^3 + \omega_{02}^3 + 2\omega_{01}^2\omega_{02} + 2\omega_{01}\omega_{02}^2 \\ &+ s_1^2\omega_{01}^2\omega_{02} + s_1^2\omega_{01}\omega_{02}^2) + \omega_1^3(\omega_{01}^4\omega_{02} + \omega_{01}\omega_{02}^4 + \\ &2\omega_{01}^3\omega_{02}^2 + 2\omega_{01}^2\omega_{02}^3 + s_1^2\omega_{01}^3\omega_{02}^2 + s_1^2\omega_{01}^2\omega_{02}^3) \\ &-\omega_1(\omega_{01}^4\omega_{02}^3 + \omega_{01}^3\omega_{02}^4) = 0 .\end{aligned} \tag{231a}$$

By direct expansion,

$$(\omega_{01}+\omega_{02})^3 = (\omega_{01}^3+3\omega_{01}^2\omega_{02}+3\omega_{01}\omega_{02}^2+\omega_{02}^3). \quad (232)$$

From equation (216),

$$(\omega_{01}+\omega_{02})^3 = \left(\frac{S\omega_2}{S_1}\right)^3. \quad (233)$$

After combining equations (232) and (233),

$$(\omega_{01}^3+\omega_{02}^3) = \left(\frac{S\omega_2}{S_1}\right)^2 \cdot \frac{S\omega_2}{S_1} - 3\omega_2^2(\omega_{01}+\omega_{02}). \quad (234)$$

By substituting for $s\omega_2/s_1$ its value $(\omega_{01}+\omega_{02})$ and rearranging terms, equation (234) reduces to

$$(\omega_{01}^3+\omega_{02}^3) = \omega_2^2(\omega_{01}+\omega_{02})\left(\frac{S^2}{S_1^2}-3\right). \quad (235)$$

By using the relationship $\omega_{01}\omega_{02}=\omega_2^2$ where desired and substituting for $(\omega_{01}^3+\omega_{02}^3)$ its value (from equation (235) above) in equation (231a), the following equation in ω_1 is obtained:

$$(\omega_1^7) - \omega_1^5\omega_2^2\left(S_1^2+\frac{S^2}{S_1^2}-1\right) + \omega_1^3\omega_2^4\left(S_1^2+\frac{S^2}{S_1^2}-1\right) - \omega_1\omega_2^6 = 0. \quad (236)$$

Equation (236) is simplified to yield the quadratic equation in s_1^2, namely:

$$(S_1^2)^2(\omega_1^2\omega_2^2)-(S_1^2)(\omega_1^2+\omega_2^2)+s^2(\omega_1^2\omega_2^2)=0. \tag{237}$$

Solution of equation (237) reveals that

$$S_1^2=\frac{1}{2}\left(\frac{\omega_1^2+\omega_2^2}{\omega_1\omega_2}\right)^2\pm\sqrt{\frac{1}{4}\left(\frac{\omega_1^2+\omega_2^2}{\omega_1\omega_2}\right)^4-4s^2} \tag{238}$$

and

$$S_1=\sqrt{\frac{1}{2}\left(\frac{\omega_1^2+\omega_2^2}{\omega_1\omega_2}\right)^2\pm\sqrt{\frac{1}{4}\left(\frac{\omega_1^2+\omega_2^2}{\omega_1\omega_2}\right)^4-4s^2}}. \tag{239}$$

The corresponding equations in terms of frequency are

$$S_1^2=\frac{1}{2}\left(\frac{f_1^2+f_2^2}{f_1f_2}\right)^2\pm\sqrt{\frac{1}{4}\left(\frac{f_1^2+f_2^2}{f_1f_2}\right)-4s^2} \tag{240}$$

and

$$S_1=\sqrt{\frac{1}{2}\left(\frac{f_1^2+f_2^2}{f_1f_2}\right)^2\pm\sqrt{\frac{1}{4}\left(\frac{f_1^2+f_2^2}{f_1f_2}\right)^4-4s^2}}. \tag{241}$$

In terms of F, s_1^2 is

$$S_1^2=\frac{1}{2}\left(\frac{10^{2F_1}+10^{2F_2}}{10^{F_1+F_2}}\right)^2\pm\sqrt{\frac{1}{4}\left(\frac{10^{2F_1}+10^{2F_2}}{10^{F_1+F_2}}\right)^4-4s^2} \tag{242}$$

and s_1 is

$$s_1 = \sqrt{\frac{1}{2}\left(\frac{10^{2F_1}+10^{2F_2}}{10^{F_1+F_2}}\right)^2 \pm \sqrt{\frac{1}{4}\left(\frac{10^{2F_1}+10^{2F_2}}{10^{F_1+F_2}}\right)^4 - 4s^2}} \quad (243)$$

The $\pm$ sign in the above equations allows a choice in the value of s_1. However, consideration of equation (132) indicates that the larger value of s_1 results in less attenuation per phase-shift section.

The solutions for ω_{01} and ω_{02}, f_{01} and f_{02} or F_{01} and F_{02} in terms of s and ω, f or F follow. Combining conditions (223) and (225) results in

$$s\omega_2 = s_1 \frac{\omega_2^2}{\omega_{02}} + s_1 \omega_{02} \quad (244)$$

which may be arranged as the quadratic equation in ω_{02}

$$\omega_{02}^2 (s_1) - \omega_{02}(s\omega_2) + s_1\omega_2^2 = 0 \quad (245)$$

from which

$$\omega_{02} = \frac{s\omega_2 \pm \sqrt{s^2\omega_2^2 - 4s_1^2\omega_2^2}}{2s_1} . \quad (246)$$

A similar development for ω_{01} yields

$$\omega_{01} = \frac{S\omega_2 \pm \sqrt{S^2\omega_2^2 - 4S_1^2\omega_2^2}}{2S_1} \tag{247}$$

from which it is deduced that

$$\omega_{01} = \omega_2\left(\frac{S - \sqrt{S^2 - 4S_1^2}}{2S_1}\right) \tag{248}$$

$$\omega_{02} = \omega_2\left(\frac{S + \sqrt{S^2 - 4S_1^2}}{2S_1}\right). \tag{249}$$

Other variations of equations (248) and (249) are

$$f_{01} = f_2\left(\frac{S - \sqrt{S^2 - 4S_1^2}}{2S_1}\right) \tag{250}$$

$$f_{02} = f_2\left(\frac{S + \sqrt{S^2 - 4S_1^2}}{2S_1}\right) \tag{251}$$

and

$$F_{01} = F_2 + \log_{10}\left(\frac{S - \sqrt{S^2 - 4S_1^2}}{2S_1}\right) \tag{252}$$

$$F_{02} = F_2 + \log_{10}\left(\frac{S + \sqrt{S^2 - 4S_1^2}}{2S_1}\right). \tag{253}$$

It has been demonstrated that a three critical frequency R-C network may be developed to yield the same frequency-phase characteristics as a two pole L-C network, provided that each individual one critical frequency R-C network has a value of s equal to the s_1 determined by any of equations (238) through (243). The critical frequency, f_{01}, of the first one critical frequency R-C network is determined by equation (250), while equation (251) determines the critical frequency, f_{02}, of the second.

A suggested method for evaluating the circuit constants for channels A and B is given below.

(1) Select a value for f_M and determine $F_M = \log_{10} f_M$.

(2) Add .225 to F_M to obtain F_{2B}; subtract .225 from F_M to obtain F_{2A}.

(3) Add .9 to F_{2A} to obtain F_{3A}; subtract .9 from F_{2A} to obtain F_{1A}.

(4) Add .9 to F_{2B} to obtain F_{3B}; subtract .9 from F_{2B} to obtain F_{1B}.

(5) Determine f_{1A}, f_{2A}, f_{3A}, f_{1B}, f_{2B}, f_{3B} from the relationship $10^F = f$.

(6) Determine s_1 for each individual one critical frequency network from either formula (239), (241) or (243). For the given conditions, $s = 22.345$, $s_1 = 7.499$.

For Channel A:

(7) Determine f_{01A}, f_{02A} from formulae (250) and (251), respectively.

For first single critical frequency R-C network of Channel A:

(8) Select a suitable value of R_{1A01}.

(9) R_{2A01} then equals R_{1A01}/a''' where a''' equals $1/(s_1+2)$.

(10) R_{3A01} then equals $R_{2A01}(1-4a''' /4a''')$.

(11) Determine C_{1A01}, C_{2A01}, and C_{3A01} from the relationships

$$\omega_{2A01} = 1/R_{1A01}C_{1A01} = 1/R_{2A01}C_{2A01} = 1/R_{3A01}C_{3A01}.$$

For second single critical frequency R-C network of Channel A:

(12) Let $R_{1A02} = R_{1A01}$, $R_{2A02} = R_{2A01}$, $R_{3A02} = R_{3A01}$.

(13) Determine C_{1A02}, C_{2A02} and C_{3A02} from the relationship

$$\omega_{2A02} = 1/R_{1A02}C_{1A02} = 1/R_{2A02}C_{2A02} = 1/R_{3A02}C_{3A02}.$$

For Channel B:

(14) Determine f_{2B01} and f_{2B02} from formulae (250) and (251), respectively.

For first single critical frequency R-C network of Channel B:

(15) Let $R_{1B01} = R_{1A01}$, $R_{2B01} = R_{2A01}$, $R_{3B01} = R_{3A01}$.

(16) Determine C_{1B01}, C_{2B01} and C_{3B01} from the relationship

$$\omega_{2B01} = 1/R_{1B01}C_{1B01} = 1/R_{2B01}C_{2B01} = 1/R_{3B01}C_{3B01}.$$

For second single critical frequency R-C network of Channel B:

(17) Let $R_{1B02} = R_{1A01}$, $R_{2B02} = R_{2A01}$, $R_{3B02} = R_{3A01}$.

(18) Determine C_{1B02}, C_{2B02} and C_{3B02} from the relationships

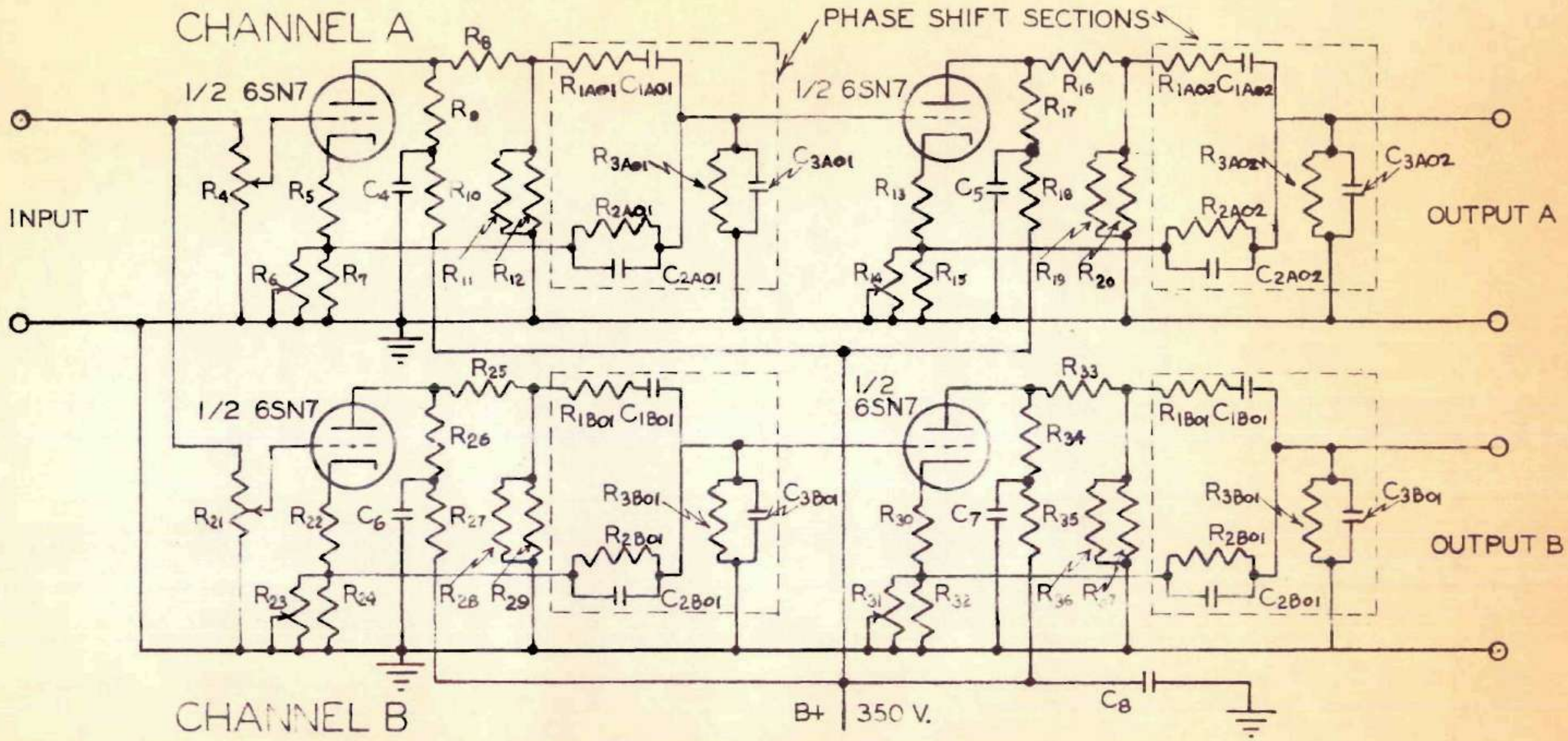
$$\omega_{2B02} = 1/R_{1B02}C_{1B02} = 1/R_{2B02}C_{2B02} = 1/R_{3B02}C_{3B02}.$$

EXPERIMENTAL RESULTS

The experimental portion of this paper is presented to verify the validity of the mathematical calculations. The cirait employed for channels A and B is shown schematically in Figure 15. The accompanying list of parts for the circuit is given in Appendix IV.

The voltage sources for each phase-shift section are obtained from the voltage drops across equal plate and cathode resistors. A shunt-feed arrangement permits each of these resistors to be connected to a common ground point. Potentiometers R_6, R_{14}, R_{23}, and R_{31} are provided for adjustment of the driving voltages. At the input to each channel, potentiometers R_4 and R_{21} control the relative level of the applied signals. The plate supply for each tube is obtained by a shunt-feed filter arrangement. A typical combination is shown by R_9, R_{10} and C_4. It was found that the filter $R_{10}C_4$ decreased interaction between the signals in each section by preventing signals from passing through the common plate supply. Condenser, C_8, also provides a bypass across the plate supply. The phase-shift sections are identical to those previously discussed under the three critical frequency R-C network. The circuit constants were evaluated according to the suggested procedure of the preceding section. A center frequency (f_M) of 900 cycles per second and a value of R_{1A01} equal to 20,000 ohms were chosen. It was found necessary to shield each phase-shift section from body capacity and stray 60 cycle interference.

The relative phase angle between two voltages was determined by applying the voltages to the horizontal and vertical channels of a cathode ray oscilloscope and observing the shape of the traces



LIST OF PARTS IS CONTAINED IN APPENDIX IV

SCHEMATIC DIAGRAM OF TWO CHANNEL
90° PHASE DIFFERENCE SYSTEM

FIG. 15

obtained. Terman[16] discusses this method of phase measurement completely in the reference given below. A variable audio-frequency generator provided the input signal. It was found necessary to employ one stage of amplification between each channel and the oscilloscope. This was expected, since each channel has a high degree of attenuation.

Test runs were performed on each individual phase-shift section, on each channel and on the two channels operating together. For the tests on the individual phase-shift sections and on each channel, the output of the circuit under test was connected directly to the input of an oscilloscope, and a reference voltage was applied directly from the signal generator to the horizontal input. In testing two channels together, the output of one channel was applied to the vertical input of the oscilloscope, while the output of the other channel was applied to the horizontal input of the oscilloscope. Sufficient amplification could not be obtained in the horizontal channel of the oscilloscope. As a consequence, a resistance-coupled amplifier, compensated to yield constant gain over the desired range of frequencies, was used between the horizontal section of the oscilloscope and the channel under consideration. An identical amplifier was then necessary between the output of the other channel and the vertical oscilloscope section in order to retain proper phase and polarity relationships at the oscilloscope terminals. The results of the phase tests on the individual channels are tabulated in Appendix V. The phase difference which was observed between both channels operating together remained

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[16] Terman, F. E., op. cit., pp. 947-949.

very close to 90° in the frequency range from 140 to 6500 cycles per second. The exact error was too slight to be measured by the method used.

The amplitude of the output of each phase-shift section and each channel, as measured by oscilloscope methods, was substantially constant.

CONCLUSIONS

The close agreement between calculated and test results indicates that the extended form of the R-C network is entirely practical for use as a producer of a 90° phase difference over a wide audio range. Further, since the calculated error of the extended network (Figure 12) is less than the calculated error of Dome's single critical frequency network (Figure 6), not only has the frequency range been increased, but the accuracy has also been improved. The methods developed in the previous sections may be applied successfully to effect further extensions of the frequency range.

When a conventional amplitude modulated system is 100 percent modulated, two-thirds of the transmitted energy is contained in the carrier wave. The remaining one-third is distributed equally between the upper and lower sidebands. It can be shown that the entire intelligence may be extracted from the carrier and one sideband. Single-sideband transmission is accomplished by transmitting only one sideband and then supplying the carrier frequency wave at the receiving end. The main advantages of single-sideband systems are the economies of transmission and the conservation of space in the frequency spectrum. In the case of 100 percent modulation and single tone signal, a saving of five-sixths of the energy which would be transmitted by conventional methods is effected. Also, a much higher signal to noise ratio may be obtained with single-sideband methods. In addition, one-half of the spectrum space which would normally be required is conserved, thus allowing two single-sideband stations to operate in the frequency range of one conventional amplitude modulated station.

The above comparisons are based on the assumption that any 90° phase difference networks employed yield exactly 90° phase difference with equal output voltages. A deviation in either of the above conditions may cause a portion of the transmitted energy to appear in an unwanted sideband.

Small errors in angle and magnitude are permissible in radio and power line carrier single-sideband systems. As an example, should the magnitude of the unwanted sideband be 5 percent of the magnitude of the desired sideband, the energy contained in the unwanted sideband is but .25 percent of the energy in the desired sideband.

However, certain telephonic applications of single-sideband transmission require much closer tolerance than that cited above. Highly selective filter systems are frequently employed to maintain the tolerance. While the calculated $90 \pm 2^\circ$ phase difference obtained from the system herein described might not be sufficiently accurate to replace filters, use of the 90° phase-shift system in conjunction with filters should materially reduce the burden on the filters, and, as a consequence, less filter capacity would be required.

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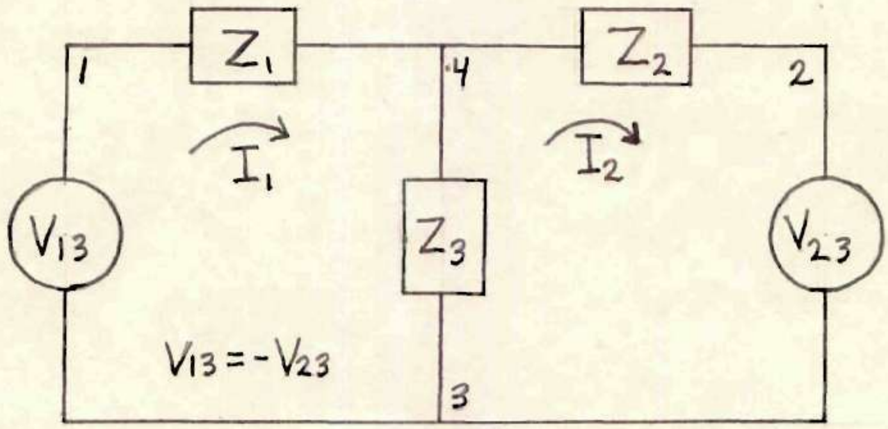
APPENDIX I

Voltage Equation for General Phase-Shift Section



The standard mesh equations for the solution of currents I_1 and I_2 are

$$V_{13} = I_1(Z_1 + Z_3) + I_2(-Z_3) \qquad (254)$$

$$-V_{23} = V_{13} = I_1(-Z_3) + I_2(Z_2 + Z_3). \qquad (255)$$

By solving equations (254) and (255) simultaneously by determinant methods,

$$I_1 = \frac{\begin{vmatrix} V_{13} & -Z_3 \\ V_{13} & Z_2 + Z_3 \end{vmatrix}}{\begin{vmatrix} Z_1 + Z_3 & -Z_3 \\ -Z_3 & Z_2 + Z_3 \end{vmatrix}} = \frac{V_{13}(Z_2 + 2Z_3)}{Z_1Z_2 + Z_1Z_3 + Z_2Z_3} \qquad (256)$$

$$I_2 = \frac{\begin{vmatrix} Z_1 + Z_3 & V_{13} \\ -Z_3 & V_{13} \end{vmatrix}}{\begin{vmatrix} Z_1 + Z_3 & -Z_3 \\ -Z_3 & Z_2 + Z_3 \end{vmatrix}} = \frac{V_{13}(Z_1 + 2Z_3)}{Z_1 Z_2 + Z_1 Z_3 + Z_2 Z_3} \quad (257)$$

$$I_1 - I_2 = \frac{V_{13}(Z_2 - Z_1)}{Z_1 Z_2 + Z_1 Z_3 + Z_2 Z_3} \quad (258)$$

$$V_{43} = \frac{V_{13}(Z_2 - Z_1) Z_3}{Z_1 Z_2 + Z_1 Z_3 + Z_2 Z_3} = \frac{V_{13}(Z_2 - Z_1)}{\frac{Z_1 Z_2}{Z_3} + (Z_1 + Z_2)} \quad (259)$$

$$V_{34} = -V_{43} = \frac{V_{13}(Z_1 - Z_2)}{\frac{Z_1 Z_2}{Z_3} + (Z_1 + Z_2)} . \quad (260)$$

APPENDIX II

Angle and Error Calculations for either One Pole L-C or One Critical Frequency R-C Network

$F_o = 2$; $s = 4$.

f in c.p.s.	F	Calc. tan θ	Calc. θ in degrees	Desired θ by Slope method	Error in degrees for Slope method	Desired θ by Dome's method	Error in degrees for Dome's method
3.98	0.6	-.327	-18.10	4.80	-22.90	11.96	-30.06
5.01	0.7	-.416	-22.56	-8.40	-14.16	-1.77	-20.79
6.31	0.8	-.542	-28.40	-21.60	-6.80	-15.48	-12.92
7.94	0.9	-.713	-35.48	-34.80	1.32	-29.19	-6.29
10.00	1.0	-.966	-44.00	-48.00	4.00	-42.90	-1.10
12.59	1.1	-1.386	-54.20	-61.20	7.00	-56.61	2.41
15.85	1.2	-2.250	-66.10	-74.40	8.30	-70.32	4.12
19.95	1.3	-5.370	-79.45	-87.60	8.15	-84.03	4.58
25.12	1.4	14.370	-93.37	-100.80	7.43	-97.74	4.37
31.62	1.5	2.882	-109.12	-114.00	4.88	-111.45	2.33
39.81	1.6	1.472	-124.20	-127.20	3.00	-125.16	0.96
50.12	1.7	0.863	-139.60	-140.40	0.80	-138.87	-0.73
63.10	1.8	0.479	-154.37	-153.60	0.23	-152.58	-1.79
79.43	1.9	0.236	-166.72	-166.80	0.08	-166.29	-0.43
100.0	2.0	0.000	-180.00	-180.00	0.00	-180.00	0.00
125.9	2.1	-0.236	-193.28	-193.20	-0.08	-193.71	0.43
158.5	2.2	-0.479	-206.63	-206.40	-0.23	-207.42	1.79
199.5	2.3	-0.863	-220.40	-219.60	-0.80	-221.13	0.73
251.2	2.4	-1.472	-235.80	-232.80	-3.00	-234.84	-0.96
316.2	2.5	-2.882	-250.88	-246.00	-4.88	-248.55	-2.33
398.1	2.6	-14.370	-266.63	-259.20	-7.43	-262.26	-4.37
501.2	2.7	5.370	-280.55	-272.40	-8.15	-275.97	-4.58
631.0	2.8	2.250	-293.90	-285.60	-8.30	-289.68	-4.12
794.3	2.9	1.386	-305.80	-298.80	-7.00	-303.39	-2.41
1000.	3.0	0.966	-316.00	-312.00	-4.00	-317.10	1.10
1259.	3.1	0.713	-326.52	-325.20	-1.32	-330.81	6.29
1585.	3.2	0.542	-331.60	-338.40	6.80	-344.52	12.92
1995.	3.3	0.416	-337.44	-351.60	14.16	-358.23	20.79
2512.	3.4	0.327	-341.90	-364.80	22.90	-371.94	30.06

APPENDIX III

Angle and Error Calculations for either Two Pole L-C or Three Critical Frequency R-C Network

$F_1 = 1.1$; $F_2 = 2.0$; $F_3 = 2.9$; $s \approx 22.342$.

f in c.p.s.	F	Calc. tan θ	Calc. θ in degrees	Desired θ in degrees	Error in degrees
1.	0.0	-0.476	-25.45	40.00	-65.45
1.59	0.2	-0.836	-39.90	0.00	-39.90
3.16	0.5	-3.484	-74.00	-60.00	-14.00
3.98	0.6	-71.850	-89.20	-80.00	-9.20
5.01	0.7	3.582	-105.60	-100.00	-5.60
6.31	0.8	1.472	-124.20	-120.00	-4.20
7.94	0.9	0.773	-142.30	-140.00	-2.30
10.00	1.0	0.343	-161.05	-160.00	-1.05
12.59	1.1	0.000	-180.00	-180.00	0.00
15.85	1.2	-0.346	-199.10	-200.00	0.90
19.95	1.3	-0.802	-218.75	-220.00	1.25
25.12	1.4	-1.667	-239.05	-240.00	0.95
31.62	1.5	-5.468	-259.65	-260.00	0.35
35.48	1.55		-270.00	-270.00	0.00
39.81	1.6	5.510	-280.30	-280.00	-0.30
50.12	1.7	1.680	-300.75	-300.00	-0.75
63.10	1.8	0.810	-321.00	-320.00	-1.00
79.43	1.9	0.345	-340.95	-340.00	-0.95
100.0	2.0	0.000	-360.00	-360.00	0.00
125.9	2.1	-0.345	-379.05	-380.00	0.95
158.5	2.2	-0.810	-399.00	-400.00	1.00
199.5	2.3	-1.680	-419.25	-420.00	0.75
251.2	2.4	-5.510	-439.70	-440.00	0.30
281.8	2.45		-450.00	-450.00	0.00
316.2	2.5	5.468	-460.35	-460.00	-0.35
398.1	2.6	1.667	-480.95	-480.00	-0.95
501.2	2.7	0.802	-501.25	-500.00	-1.25
631.0	2.8	0.346	-520.90	-520.00	-0.90
794.3	2.9	0.000	-500.00	-500.00	0.00
1000.	3.0	-0.343	-558.95	-560.00	1.05
1259.	3.1	-0.773	-577.70	-580.00	2.30
1585.	3.2	-1.472	-595.80	-600.00	4.20
1995.	3.3	-3.582	-614.40	-620.00	5.60
2512.	3.4	71.850	-630.80	-640.00	9.20
3162.	3.5	3.484	-646.00	-660.00	14.00
6310.	3.8	0.836	-680.10	-720.00	39.90
10000.	4.0	0.476	-694.55	-760.00	65.45

APPENDIX IV

Circuit Constants for Experimental Two Channel 90^{o} Phase Difference System Shown in Figure 15

Circuit Constant	Value	Tolerance
R5,R13,R22,R30	100 ,1w.	10%
R7,R12,R15,R20,R24,R29,R32,R37	220 ,1w.	"
R9,R10,R17,R18,R26,R27,R34,R35	10,000 ,2w.	"
R8,R16,R25,R33	10,000 ,1w.	"
R11,R19,R28,R36	5,000 ,1w.	"
R4,R21	50,000 ,1w. pot.	"
R6,R14,R23,R31	5,000 ,1w. pot.	"
C4,C5,C6,C7	8uf.,350v.	"
C8	50uf.,350v.	"
Channel A:		
R1A01,R2A02*	20,000	1%
R2A01,R2A02*	189,180	"
R3A01,R3A02*	260,072	"
C1A01**	.038532uf.	"
C2A01**	.004074uf.	"
C3A01**	.002963uf.	"
C1A02**	.005724uf.	"
C2A02**	.000605uf.	"
C3A02**	.000440uf.	"
Channel B:		
R1B01,R1B02*	20,000	1%
R2B01,R2B02*	189,180	"
R3B01,R3B02*	260,072	"
C1B01**	.013672uf.	"
C2B01**	.001445uf.	"
C3B01**	.001051uf.	"
C1B02**	.002031uf.	"
C2B02**	.000215uf.	"
C3B02**	.000156uf.	"

* Resistors for Channels A and B were adjusted at d.c. values on a Wheatstone Bridge whose accuracy was 1%.

** Condensers for Channels A and B were adjusted to have voltage drops equal to voltage drops across the corresponding resistors at critical frequency.

APPENDIX V

Network Test Data

Channel	Phase-shift section	Phase angle (in degrees)	Observed frequency (in c.p.s.)	Calculated frequency (in c.p.s.)
A	First	90	25.5	27.
"	"	180	206.	207.
"	"	270	1550.	1581.
"	Second	90	193.	181.7
"	"	180	1393.	1390.
"	"	270	10000.	10640.
"	Both	90	21.	23.94
"	"	180	69.	67.45
"	"	270	198.	191.00
"	"	360	540.	535.80
"	"	450	1510.	1510.
"	"	540	4200.	4256.
"	"	630	13500.	12000.
B	First	90	77.	76.15
"	"	180	600.	582.
"	"	270	4800.	4460.
"	Second	90	520.	511.
"	"	180	3980.	3917.
"	"	270	35400.	29910.
"	Both	90	61.	67.45
"	"	180	194.	191.
"	"	270	550.	535.8
"	"	360	1600.	1510.
"	"	450	4420.	4256.
"	"	540	13200.	12000.
"	"	630	45000.	33830.